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TECHNICAL NOTE

Calculation of the radii of curvature of the crystalline lens surfaces

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Summary

A new computing scheme was developed for calculating the radius of curvature of the anterior and posterior surfaces of the crystalline lens from the measured heights of the Purkinje images. The scheme can be applied to objects at any distance from the corneal vertex, for both a *stationary* object mounted independently of the camera and for a *mobile* object attached to the camera where the distance of the object to the corneal vertex will change as the camera is refocused from image PI to PIII and PIV. The method can also be used if different objects are employed to form each Purkinje image. The scheme also avoids the need to collimate objects in order to employ the equivalent mirror theorem or to calibrate phakometers with known spherical surfaces where exact relationships are unknown. Copyright © 1996 The College of Optometrists. Published by Elsevier Science Ltd.

Introduction

The usual technique for measurement of the radii of curvature of the surfaces of the crystalline lens is phakometry, in which the height of the Purkinje images formed by the anterior surface of the cornea and from the anterior and posterior surfaces of the lens are recorded using photographic or video methods. If the radius of curvature of the cornea is known the radii of curvature of the lens surfaces can be calculated using the equivalent mirror theorem (Sorsby *et al.*, 1961; Bennett and Rabbetts, 1989; Henson, 1991). The equivalent mirror theorem states that an optical system consisting of a number of refracting surfaces followed by a reflecting surface can be replaced by a single 'equivalent' mirror, and the radius of curvature of the equivalent mirror r_3' for the anterior surface determined from the expression

$$r_3' = r_1 \left(\frac{h_3'}{h_1'} \right) \tag{1}$$

where r_1 is the radius of curvature of the cornea and h_1' and h_3' are the heights of the Purkinje image PI and PIII respectively. The radius of curvature for the equivalent mirror for the posterior surface given by

$$r_4' = r_1 \left(\frac{h_4'}{h_1'} \right) \tag{2}$$

where h_4' is the height of the PIV Purkinje image. The real radii of curvature can then be calculated.

The difficulties of the technique are well known (Henson, 1991; Van Heen and Goss, 1988; Mutti *et al.*, 1992). For the method to be valid, the objects must be at optical infinity and the Purkinje images are formed in different planes making clear images difficult with a single frame. If separate records are made for Purkinje images PI/PIV and PIII, the heights of the images will also depend on whether the object is attached to the camera (*mobile* object), or mounted independently (*stationary* object). The problems have been addressed by collimating objects (Mutti *et al.*, 1992), calibrating instruments with steel spheres of known radius of curvature (Mutti *et al.*, 1992), by choosing an intermediate focal plane for the recording device (Van Heen and Goss, 1988), by refocusing the camera by changing the camera distance from the eye, by using a telecentric stop (Phillips *et al.*, 1988), or by calculation (Smith and Garner, 1996). While it may be possible to derive an exact method to calculate both the anterior and posterior radii of curvature of the lens for all object heights and positions along the lines of the method previously described for the anterior

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radius of curvature (Smith and Garner, 1996), there are some difficulties in extending that method, and at present no exact method exists for the general case described above. If finite target distances are to be used in derivation of the radii of curvature of the lens, then targets must be collimated or Purkinje image heights for the equivalent mirror must be calibrated against spheres of known radius of curvature.

The purpose of this study was to develop an iterative procedure for calculation of the radius of curvature of both the anterior and posterior surfaces of the crystalline lens that provides a general solution to the difficulties outlined above. The computing scheme can be applied to instruments with finite object distances, with objects of different heights and with *stationary* or *mobile* objects.

Method

The paraxial transfer and refraction equations presented here are well known (Kingslake, 1978) and not newly developed, but have been presented in a form that can be readily applied to the technique of phakometry. The Gullstrand-Emsley schematic eye (Bennett and Rabbetts, 1989) was used in the following derivations of the equations and in the example of the computing scheme. Although this eye is a three-surface eye, with the cornea represented as a single refracting surface, it is an appropriate model for oculometric studies as measurement of the posterior surface of the cornea is not usually made. However the scheme could be readily expanded to include a cornea of finite thickness and posterior refracting surface. It is assumed that the usual biometric data is available including corneal radius of curvature, anterior chamber depth, lens thickness and vitreous chamber depth.

In general, if an object height h is placed at a distance l in front of a spherical surface of radius of curvature r separating media with refractive indices of n_1 in object space and n_2 in image space, then the image distance l' for the image formed by refraction will be given by

$$l' = \frac{n_2 l r}{(n_1 r + l(n_2 - n_1))} \tag{3}$$

while the reflected image can be found by substituting $n_2 = -n_1$ to give

$$l' = \frac{lr}{(2l - r)} \tag{4}$$

These basic equations are used to find the image positions and heights for Purkinje images PI, PIII and PIV in terms of the schematic eye shown in *Figure 1*.

In the case of all three Purkinje images, it is assumed that the camera has a fixed focus distance of w, and that when the camera is focused on the corneal apex, the distance l_0 from the object to the corneal apex is known. That is, the distance w is held constant and focusing on the images is achieved by moving the camera alone. It is also assumed that the object used to form the Purkinje image has a fixed relationship to the camera when a *mobile* object is used. A diagram showing the relationship between the essential components for a *stationary* object forming image PIII is given in *Figure 2*. As the object is fixed,

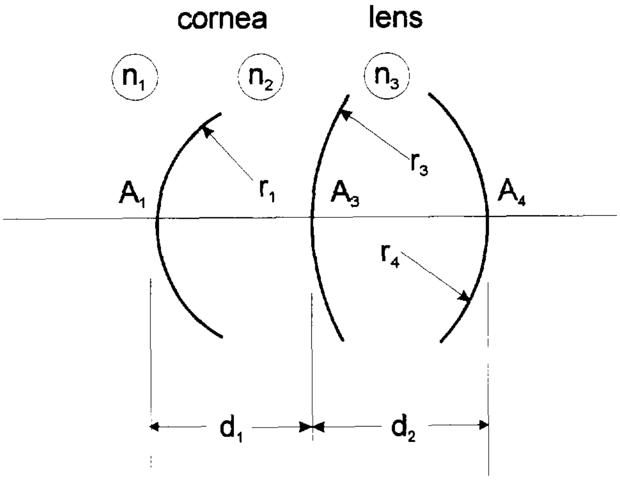

Figure 1. Three-surface schematic eye employed for calculation of Purkinje image heights.

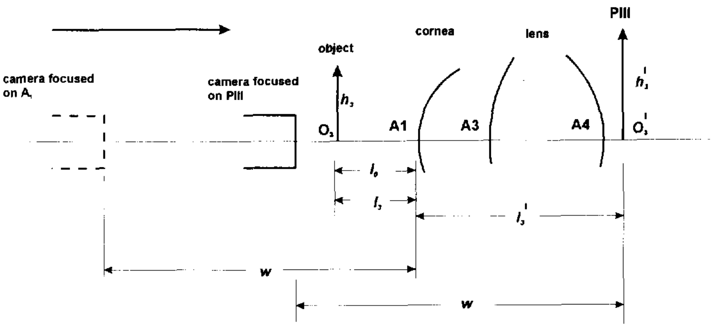

Figure 2. A *stationary* object at O_3. The camera focus is fixed at a distance w and the Purkinje image PIII is formed at a distance l'_3 from the corneal vertex.

the object distance l_3 will be equal to l_0.

The situation where a *mobile* object is used for the formation of image PIII is shown in *Figure 3*. Before the camera is moved forward to focus on the image, the distance from the camera to the corneal apex w will be given by

$$w = l_0 + x \tag{5}$$

where x is the distance from the camera to the object.

After the camera has moved to focus on the image, the distance w will be given by

$$w = l_3' + l_3 + x \tag{6}$$

and

$$l_0 = l_3 - l_3' \tag{7}$$

A similar relationship will hold for each Purkinje image where a *mobile* object is used.

Purkinje image PI

For a *stationary* object the object distance l_1 will be equal to l_0.

For a *mobile* object, the object and image distances can be found using an iterative procedure, assuming the initial object distance is given by l_0 and a first approximation to the image distance given by

$$l_1' = \frac{l_1 r_1}{2l_1 - r_1} \tag{8}$$

A second estimate of l_1 can be made from the expression

$$l_1 = l_0 + l_1' \tag{9}$$

and an iteration performed for Equations (8) and (9) until Equation (10) is satisfied,

$$l_0 = l_1 - l_1' \tag{10}$$

thus giving the final object and image distances for the particular radius of curvature of the cornea.

The magnification will be given by

$$M_1 = \frac{l_1'}{l_1} \tag{11}$$

and the height of the image PI will be

$$h_1' = h_1 M_1 \tag{12}$$

where h_1 is the height of the object used to form the PI image.

Purkinje image PIII

For a *stationary* object the object distance l_3 will be equal to l_0 as shown in *Figure 2*. For a *mobile* object, the object and image distances will change with the movement of the camera as shown in *Figure 3*, but Equation (7) must always be satisfied. The first step was to determine l_3 and l_3' for a particular value for r_3 using an iterative procedure with l_0 as a first approximation for l_3. The raytrace proceeds as shown below with an initial estimate for r_3, the radius of curvature of the anterior surface of the crystalline lens.

Refraction at the cornea

$$l_{31} = \frac{n_2 l_3 r_1}{r_1 + l_3(n_2 - n_1)} \tag{13}$$

Transfer to the lens

$$l_{32} = l_{31} - d_1 \tag{14}$$

Reflection at the lens

$$l_{33} = \frac{-l_{32} r_3}{r_3 - 2l_{32}} \tag{15}$$

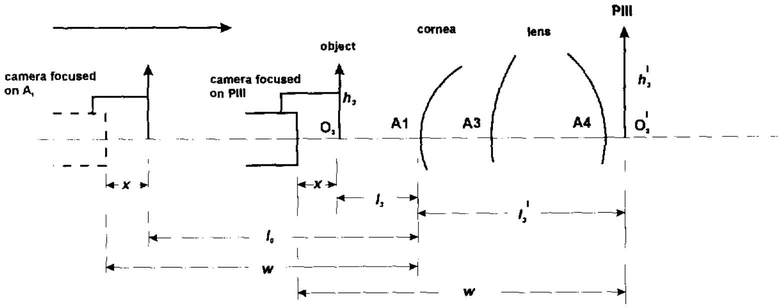

Figure 3. A *mobile* object attached to the camera with a longitudinal displacement x. The camera has a fixed focus w, and when focused on the corneal apex the target is a distance l_0 from the cornea. When the camera is focused on image PIII, the object is a distance l_3 from the cornea.

Transfer to the cornea

$$l_{34} = l_{33} + d_1 \tag{16}$$

Refraction at the cornea

$$l'_3 = \frac{n_1 l_{34} r_1}{n_2 r_1 + l_{34}(n_1 - n_2)} \tag{17}$$

A second approximation to the object distance is given by

$$l_3 = l_0 + l'_3 \tag{18}$$

and an iteration through Equations (13) to (18) is performed until Equation (19) is satisfied.

$$l_0 = l_3 - l'_3 \tag{19}$$

For the initial value of r_3 chosen the magnification will be

$$M_3 = \frac{l_{31} l_{33} l'_3}{l_3 l_{32} l_{34}} \tag{20}$$

and the calculated Purkinje image height given by

$$h'_3 = h_3 M_3 \tag{21}$$

where h_3 is the height of the object used to form the PIII image, and the merit function is given by

$$fn_3 = \left[\frac{h'_3}{h'_1}\right]^2 - \left[\frac{h''_3}{h''_1}\right]^2 \tag{22}$$

where h''_1 and h''_3 are the measured heights of the PI and PIII Purkinje images respectively. The use of a merit function in this scheme is akin to the use of similar functions in optimization methods for optical design which minimize the sum of squares of the aberration residuals (Kingslake, 1978).

The iteration proceeds for Equations (13) to (22), with incremental values for r_3 until the merit function fn_3 is satisfied. The number of cycles will depend on the initial value chosen for r_3, the size of the increments and the number of significant figures required for the final determination. The number of cycles required could be quite large, but does not pose a problem for modern computers. Typically, a 486 PC may take two to three seconds to minimize the merit function, depending on the increments defined.

Purkinje image PIV

In a similar manner, the object distance l_4 will be equal to l_0 for a *stationary* object, or to a first approximation to l_4 for a *mobile* object.

Refraction at the cornea

$$l_{41} = \frac{n_2 l_4 r_1}{n_1 r_1 + l_4(n_2 - n_1)} \tag{23}$$

Transfer to anterior lens

$$l_{42} = l_{41} - d_1 \tag{24}$$

Refraction at anterior lens

$$l_{43} = \frac{n_3 l_{42} r_3}{n_2 r_3 + l_{42}(n_3 - n_2)} \tag{25}$$

Transfer to posterior lens

$$l_{44} = l_{43} - d_2 \tag{26}$$

Reflection at posterior lens

$$l_{45} = \frac{-l_{44} r_4}{r_4 - 2l_{44}} \tag{27}$$

Transfer to anterior lens

$$l_{46} = l_{45} + d_2 \tag{28}$$

Refraction at anterior lens

$$l_{47} = \frac{n_2 l_{46} r_3}{n_3 r_3 + l_{46}(n_2 - n_3)} \tag{29}$$

Transfer to cornea

$$l_{48} = l_{47} + d_1 \tag{30}$$

Refraction at cornea

$$l'_4 = \frac{n_1 l_{48} r_1}{n_2 r_1 + l_{48}(n_1 - n_2)} \tag{31}$$

A second approximation to the object distance is given by

$$l_4 = l_0 - l'_4 \tag{32}$$

and an iteration through Equations (23) to (32) is performed until Equation (33) is satisfied.

$$l_0 = l_4 + l'_4 \tag{33}$$

For the initial value chosen for r_4, the magnification will be

$$M_4 = \frac{l_{41} l_{43} l_{45} l_{47} l'_4}{l_4 l_{42} l_{44} l_{46} l_{48}} \tag{34}$$

the height of image PIV

$$h'_4 = h_4 M_4 \tag{35}$$

where h_4 is the height of the object used to form the PIV image

and the merit function

$$fn_4 = \left[\frac{h'_4}{h'_1}\right]^2 - \left[\frac{h''_4}{h''_1}\right]^2 \tag{36}$$

The computing scheme outlined above includes general equations that can be applied to phakometry using *stationary* or *mobile* objects, or objects of different size and position to form each Purkinje image.

Results and discussion

An example of the scheme for the Gullstrand-Emsley schematic eye is given below, and the details of this schematic eye as proposed by Bennett and Rabbetts, 1989, are given in *Table 1*.

An example giving the endpoint for the major variables using this method is shown in *Table 2*. The object chosen for this example had a height of 15 mm at a distance of 50 mm from the cornea, although different objects at different distances could have been selected. In accordance with the sign convention, target distances to the left of the corneal vertex will be negative and distances to the right of the corneal vertex will be positive. The value for the ratio h'_3/h'_1 in this case was 1.967 and for h'_4/h'_1 was −0.791 for a stationary object and the corresponding values for a mobile target were 2.204 and −0.791. In practice, these ratios would be employed in the respective merit functions together with the measured ratios.

Table 1. The modified Gullstrand-Emsley schematic eye as proposed by Bennett and Rabbetts, 1989

Quantity	*Symbol*	*Value*
Radii of curvature		
Cornea	r_1	7.80
Crystalline lens (anterior)	r_3	11.00
(posterior)	r_4	−6.476
Axial separations		
Anterior chamber depth	d_1	3.60
Lens thickness	d_2	3.70
Refractive indices		
Aqueous humour	n_2	1.336
Crystalline lens	n_3	1.422
Vitreous humour	n_4	1.336

All dimensions in mm.

Table 2. Computing scheme for an object 15 mm high, 50 mm in front of the cornea of the modified Gullstrand-Emsley schematic eye as proposed by Bennett and Rabbetts, 1989

Quantity	Stationary *object*	Mobile *object*
l_1	−50.000	−46.402
l'_1	3.618	3.598
M_1	−0.0724	−0.0775
h'_1	−1.083	−1.163
l_3	−50.000	−39.655
l'_3	10.596	10.345
M_3	−0.142	−0.171
h'_3	−2.134	−2.563
h'_3/h'_1	1.967	2.204
l_4	−50.000	−46.313
l'_4	3.700	3.687
M_4	0.057	0.061
h'_4	0.857	0.920
h'_4/h'_1	−0.791	−0.791

All distances and image heights in mm.

The basis of this computing scheme is to equate the ratio of the measured heights of the Purkinje images PIII and PI with the ratio of the calculated image heights for these images for determination of r_3, and the corresponding ratios for determination of r_4. In each case, these ratios will be equal when the values chosen for the anterior and posterior radii of curvature for calculated ratio correspond to the measured ratio, within the assumptions of the schematic eye model and the accuracy of the measurement of the ocular dimensions. The method described allows calculation of the radius of curvature of both surfaces of the lens for various experimental arrangements and avoids the need to collimate objects (Mutti *et al.*, 1992) that are placed at finite distances in front of the cornea where conventional methods of calculation are to be used. While the scheme is somewhat less elegant than a method previously presented (Smith and Garner, 1996) for determination of the radius of curvature of the anterior surface of the lens, the present method does have some advantages. A different object height and position may be used to form the P1 Purkinje image from that used for the other images. One of the difficulties of using a single object is that the PI image is much brighter than the other two images and this may cause some saturation of the recording device, particularly if a video medium is used. The use of different objects with different object heights and intensities along the lines of the Tscherning or comparison technique (Henson, 1991) overcomes this problem and for this reason provision has been made to use three distinct objects, each with a different height and position. The computing scheme also allows calculation of the radius of curvature of the posterior surface of the lens.

It is generally accepted that if finite objects are to be used and radii of curvature calculated by employing the equivalent mirror theorem, errors will be reduced if a *mobile* rather than a *stationary* object is used (Mutti *et al.*, 1992). The reason for this is that the equivalent mirror method as usually employed is only valid for objects at optical infinity; however, if the object is attached to the recording device, the error introduced is compensated to some extent by the change in position of the object. With this computing scheme, there is no real advantage in using a *mobile* object.

In the video phakometer described by Mutti *et al.* (1992) collimated objects were attached to the camera in order to overcome the errors from finite sources. However, there are advantages in having objects at finite distances in front of the cornea, the most important of which is that observation and recording of image PIII is facilitated. The quality of image PIII is notoriously poor, due in part to the nature of the anterior lens surface but also to difficulties in correctly positioning the object and providing sufficient illuminance from the object to make the PIII image clearly visible. An object relatively close to the eye can improve the quality of this image.

Summary

A general computing scheme for calculating the radius of curvature of the anterior and posterior surfaces of the crystalline lens based on the well known paraxial transfer and refraction equations is presented. The scheme proposed allows for the object in phakometry to be either attached to the camera or mounted independently, to be at a finite distance in front of the eye, or the use of separate objects of different brightness and different distances from the eye.

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