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Understanding the Zernike Expansion for Ocular Wavefront Error Representation

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I wish to thank Larry Thibos for his helpful comments.

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The recommended standard for expressing ocular wavefront error (aberrations) can be found as Appendix 1 in the book on reserve entitled Customized Corneal Ablation: The Quest for Super Vision

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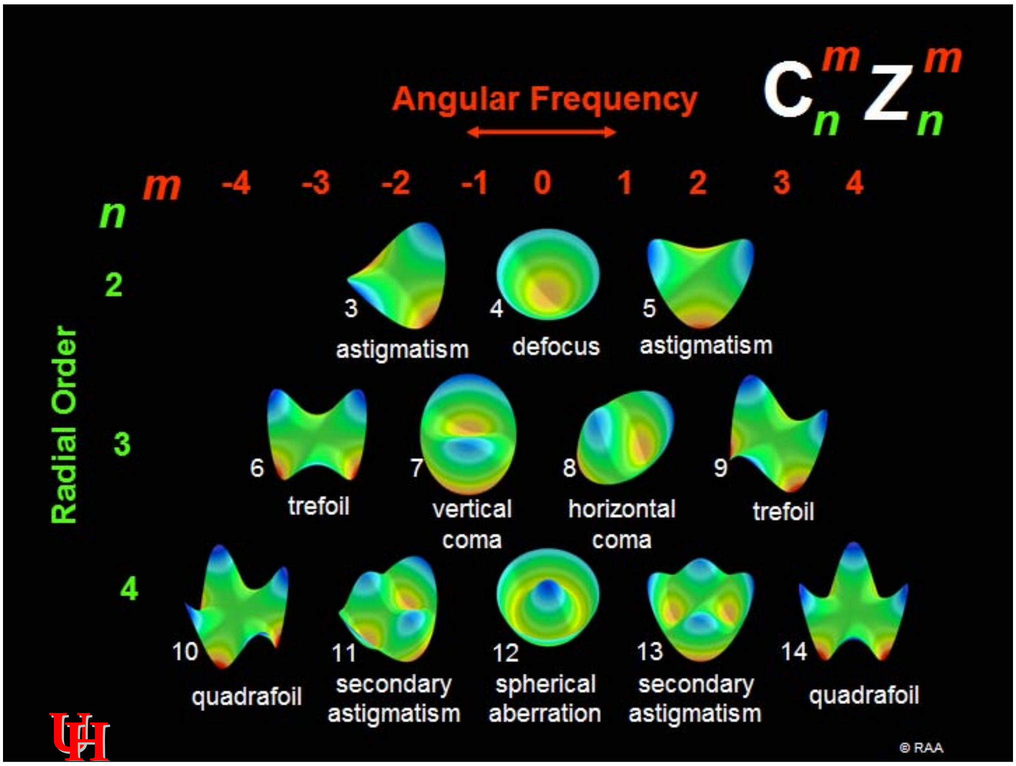


This is your road map to the Zernike expansion. Using the Zernike expansion to represent the wavefront error of the eye parcels the eye's wavefront error into unique building blocks that we can talk about.

There are several features that you should notice in this road map that the next series of slides will help explain.

In overview, each color map above is a specific mode in the Zernike expansion through the 4th radial order. There are an infinite number of modes. Notice that each mode's location in the tree is designated by a radial order (green numbers) and an angular frequency (red numbers). When referring to a specific mode a double index system is used (see the upper right corner of the slide). The Z is short hand method symbolizing all Zernike modes. The subscript and superscript designate which specific mode we are talking about. The radial order of the mode is the subscript and the angular frequency is the superscript. For example Z subscript 4 superscript 0 refers to term number 12 (single index notation) spherical aberration.

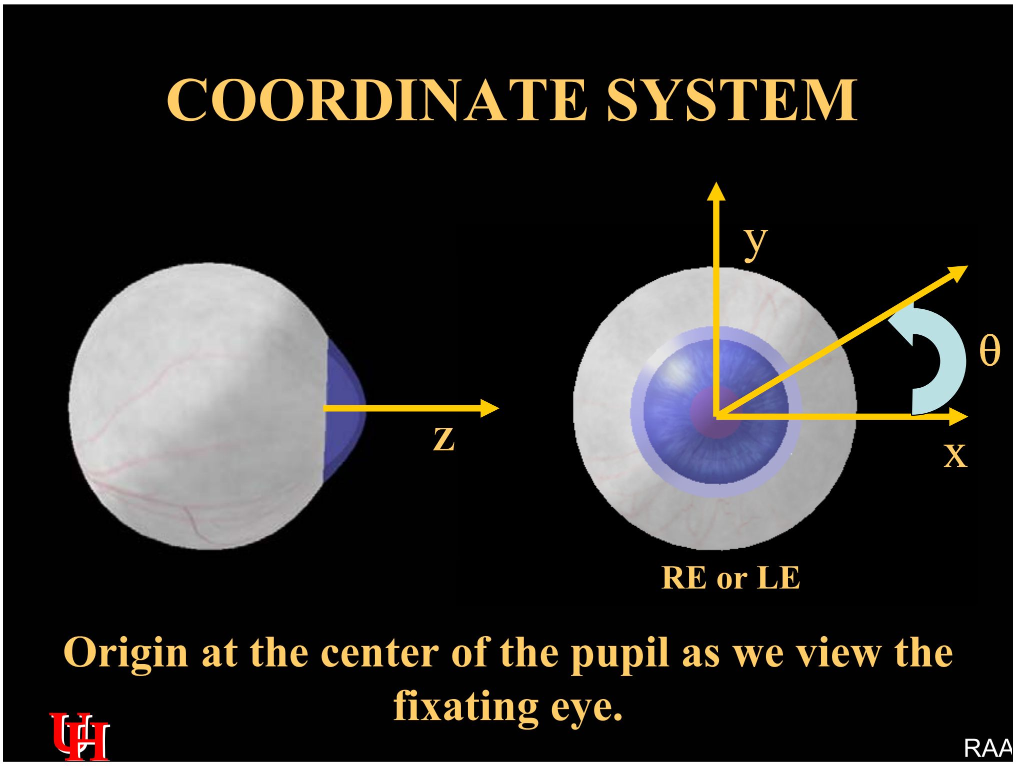
The C is short hand for the coefficient for each mode. The coefficient reveals how much of each mode is contributing to the total wavefront error.

The double index notation is preferred because it minimizes the chance for confusion. The single index notation (the white numbers beside each mode's picture) is handy when making some types of plots. As you will see the single index notation does not carry nearly as much information.

The 2nd radial order defines the aberrations associated with defocus and astigmatism. In ophthalmic terms, sphere, cylinder and axis. The 3rd order aberrations and above are collectively referred to as the higher order aberrations.

**Notes. The shapes are a true representation of the Zernike modes. However, by convention, we wish to set the wavefront error to 0 in the center of the pupil. CTView does this for all modes except those with zero angular frequency for 3D plots. CTView also has the color codes reversed in the three-D drawings from the 2-D drawings. One of these will be wrong when the ANSI Standards committee adopts a standard. For now just realize that the color code of the 3D representations in this review are opposite to those in the 2D view.

On the other hand, as drawn the figure does illustrate the fact that the mean of any Zernike mode is zero, which will not be the case if we adopt the convention that the wavefront error is zero in the center of the pupil. So for illustrating Zernikes the figure is fine but for illustrating the upcoming standard (if adopted) it will be important to add piston to make the center value of the 0 angular frequency terms 0.



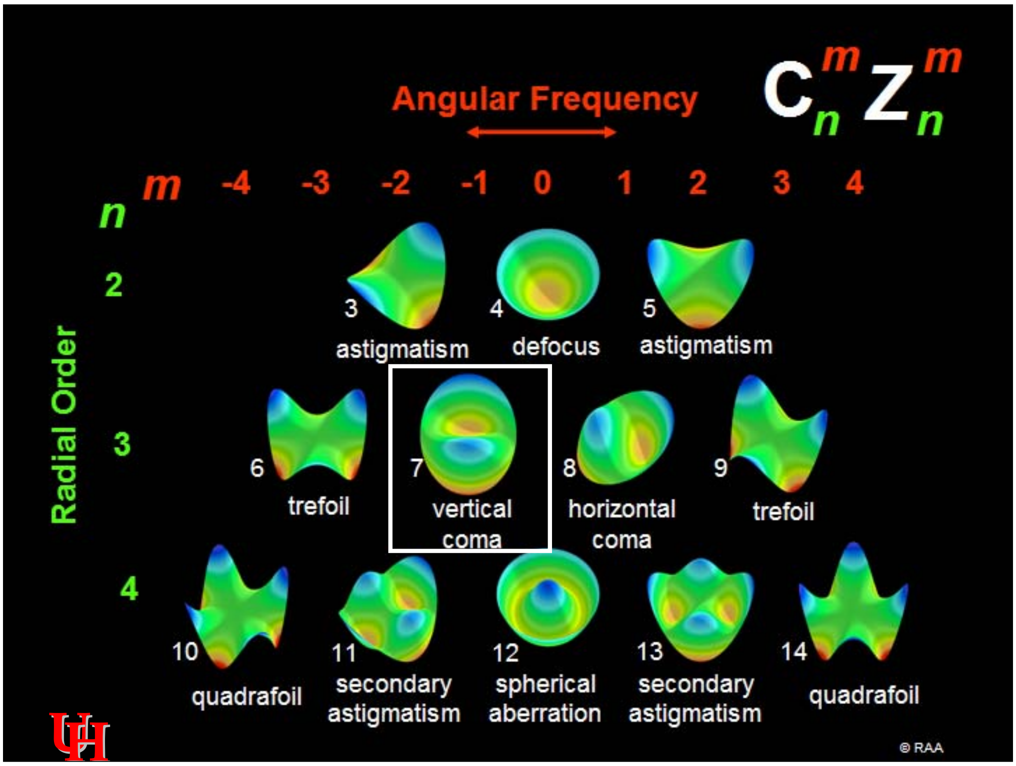
The Coordinate system.

To talk to each other and not be confused by what we mean, it is important to agree on a coordinate system for expressing wavefront error.

In traditional Cartesian coordinates as you look at the patient the origin for coordinate system for expressing the wavefront error of the eye is in the center of the entrance pupil. Positive x values are to the right. Positive y values are up and positive z values are out of the eye.

X and y values specify pupil location. Z values specify wave front error and any given pupil location.

Location within the pupil can also be specified in polar coordinates where angles are specified with zero being to the right as you look at the patient and measured in a counter clockwise direction through 360º. The angle is generally referred to as the angle θ. Locations within the pupil are then specified as an angle and a vector length from the origin along the specified angle.



Let's examine the anatomy of one of the Zernike modes in detail.

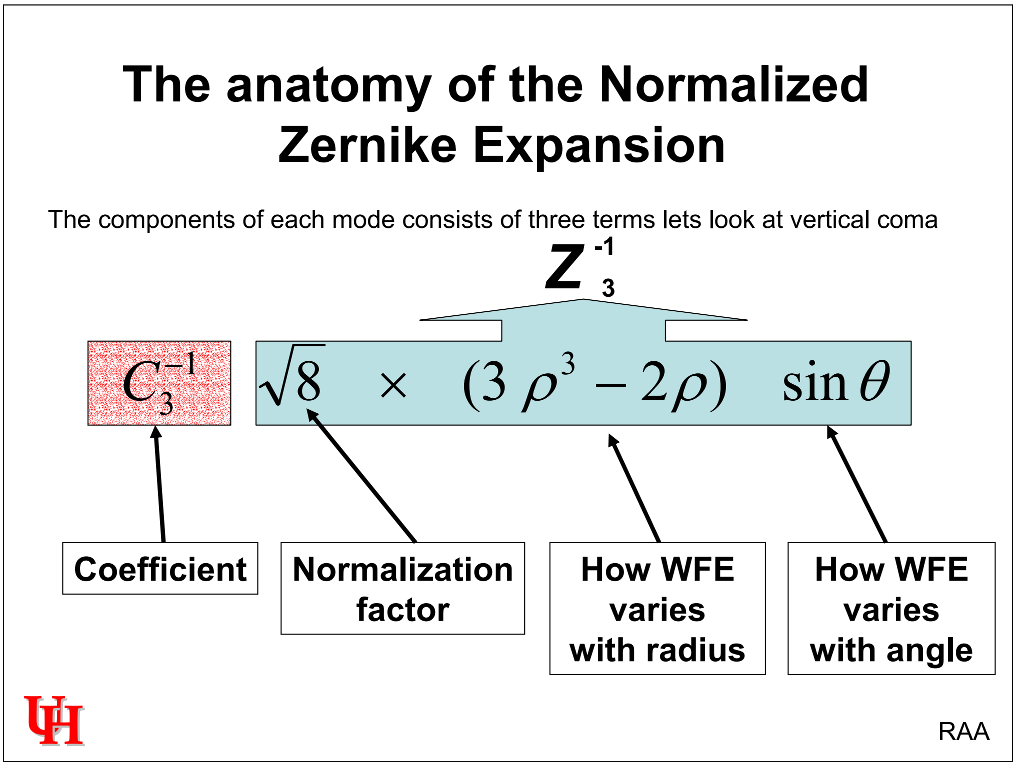
The Zernike expansion exists within a unit circle. That is the radius (ρ) varies between 0 and 1.

This is not to say that pupil size is not important. Pupil size scales the end result. For example when plotting the WFE one would multiply the maximum pupil size by by ρ and plot accordingly.

Another important feature of the Zernike expansion is that the mean value for any mode is zero.

Notice that 3 terms are missing from the pyramid illustrated above. These are terms using the single index notation 0 (the top of the pyramid- 0 radial order 0 angular frequency), 1 and 2 (the first radial order). They are omitted here because term 0 (called piston) adds a constant value of WFE to ever pupil location. Consequently it does not affect optical quality. Terms 1 (vertical prism) and 2 (horizontal prism) moves the image location and does not affect monochromatic image quality.

Now let's examine an individual Zernike mode to see its underlying anatomy. I have picked Z subscript 3 super script -1, vertical coma. Vertical coma is in the 3rd radial order and has an angular frequency of -1. In the single index notation, vertical coma is assigned the number 7.



The entire equation for how vertical coma varies within the unit pupil is made up of two major components. The coefficient (shaded in red) and the radial polynomial (shaded in light blue). The same components make up each mode as can be seen in the next slide which details the Zernike expansion for all modes through the 4th radial order.

The coefficient for each mode reveals how much of that particular mode is in the total wavefront error. It is worth noting that this is only true when the radial polynomial is normalized. Normalizing the radial polynomial is the recommended standard for specifying ocular aberrations.

There are three major components in the radial polynomial (blue highlighted portion of the equation above).

The first is a normalizing factor. The normalizing factor is a fixed number for any particular mode. For vertical coma the factor is the square root of 8. The normalizing factor is important because each mode will have unit variance when the coefficient is 1. When the mode is normalized the value assigned to the coefficient during the fitting of the wavefront error reveals the relative contribution of that particular mode to the total wavefront error. Thus, when normalized, the largest coefficient from all modes used to fit the WFE identifies the mode that is contributing the most to the total wavefront error.

The next two terms in the equation mathematically describes how the wavefront error (WFE) for vertical coma varies within the pupil. The first of these remaining two terms describes how WFE varies with pupil radius (ρ). The second describes how WFE varies with the angle (θ).

A list of the radial ploynomial for each mode is provided in the next slide.

N	n	m	Radial Polynomial	Description
0	0	0	1^(1/2) (1)	Piston
1	1	-1	4^(1/2) (1ρ) sin(θ)	Prism
2	1	1	4^(1/2) (1 ρ) cos(θ)	Prism
3	2	-2	6^(1/2) (1 ρ ^2) sin(2 θ)	Astigmatism
4	2	0	3^(1/2) (2 ρ ^2 -1)	Defocus
5	2	2	6^(1/2) (1 ρ ^2) cos(2 θ)	Astigmatism
6	3	-3	8^(1/2) (1 ρ ^3) sin(3 θ)	Trefoil
7	3	-1	8^(1/2) (3 ρ ^3 -2p) sin(θ)	Coma
8	3	1	8^(1/2) (3 ρ ^3 -2p) cos(θ)	Coma
9	3	3	8^(1/2) (1 ρ ^3) cos(3 θ)	Trefoil
10	4	-4	10^(1/2) (1 ρ ^4) sin(4 θ)	Quadrafoil
11	4	-2	10^(1/2) (4 ρ ^4 -3 ρ ^2) sin(2 θ)	Secondary Astig.
12	4	0	5^(1/2) (6 ρ ^4 -6 ρ ^2 +1)	Spherical
13	4	2	10^(1/2) (4 ρ ^4 -3 ρ ^2) cos(2 θ)	Secondary Astig.
14	4	4	10^(1/2) (1 ρ ^4) cos(4 θ)	Quadrafoil

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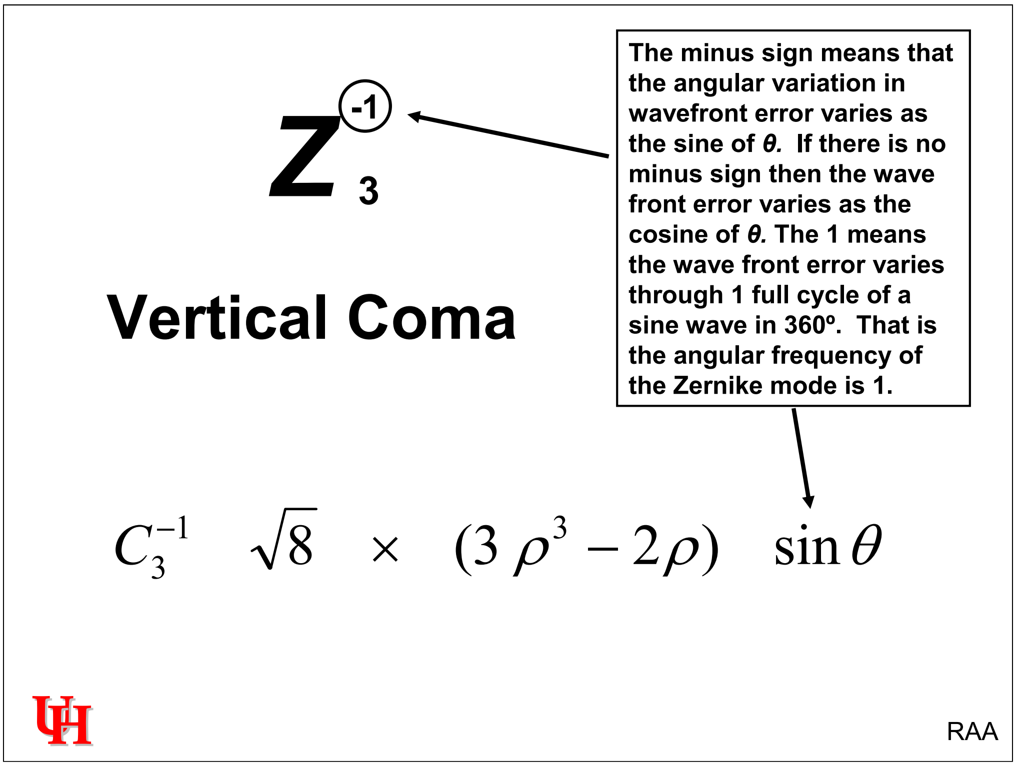
This slide illustrates the Single Index Designations (N), the Radial Order Designations (n), the Angular Frequency Designations (m), the Radial Polynomial and the Description of the Zernike expansion through the 4^{th} radial order.

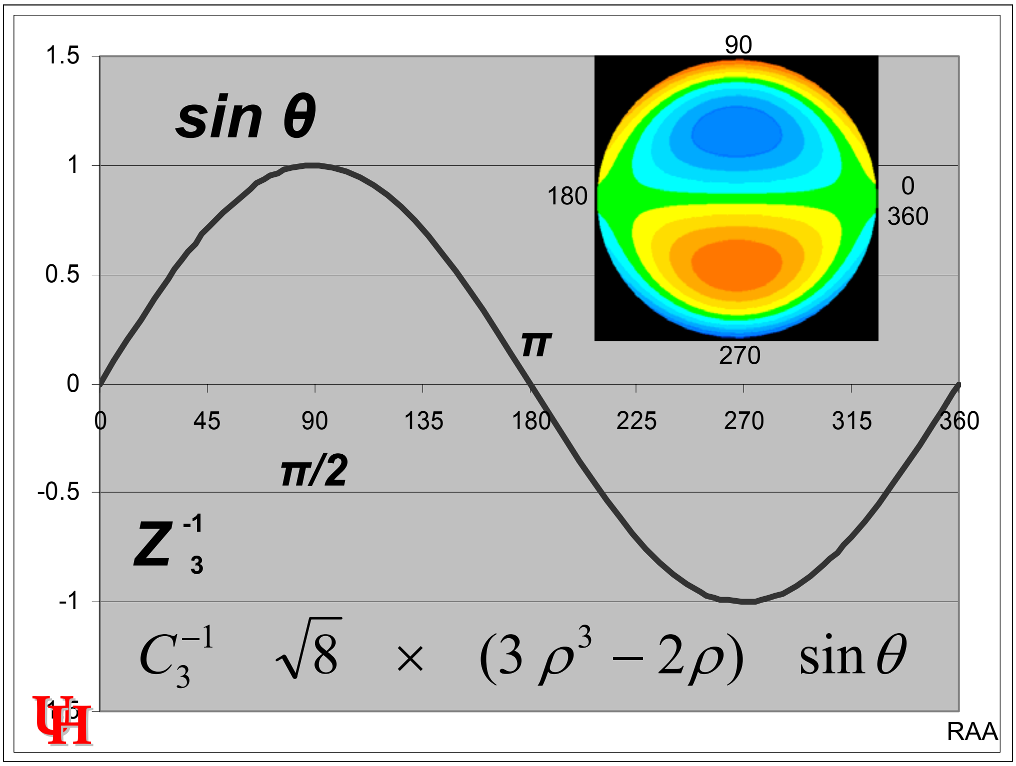
Looking back at the first slide, one can see how each of these radial polynomials affect WFE within the pupil.



Now we are in a position to more carefully examine the information contained in the double index notation.

In the superscript, the minus sign means that the angular variation in wavefront error varies as the sine of *θ*. If there is no minus sign then the wave front error varies as the cosine of *θ*. The 1 means the wave front error varies through 1 full cycle of a sine wave in 360º. That is the angular frequency of the Zernike mode is 1.

The next slide illustrates how an angular frequency of 1 influences wavefront error within the pupil.



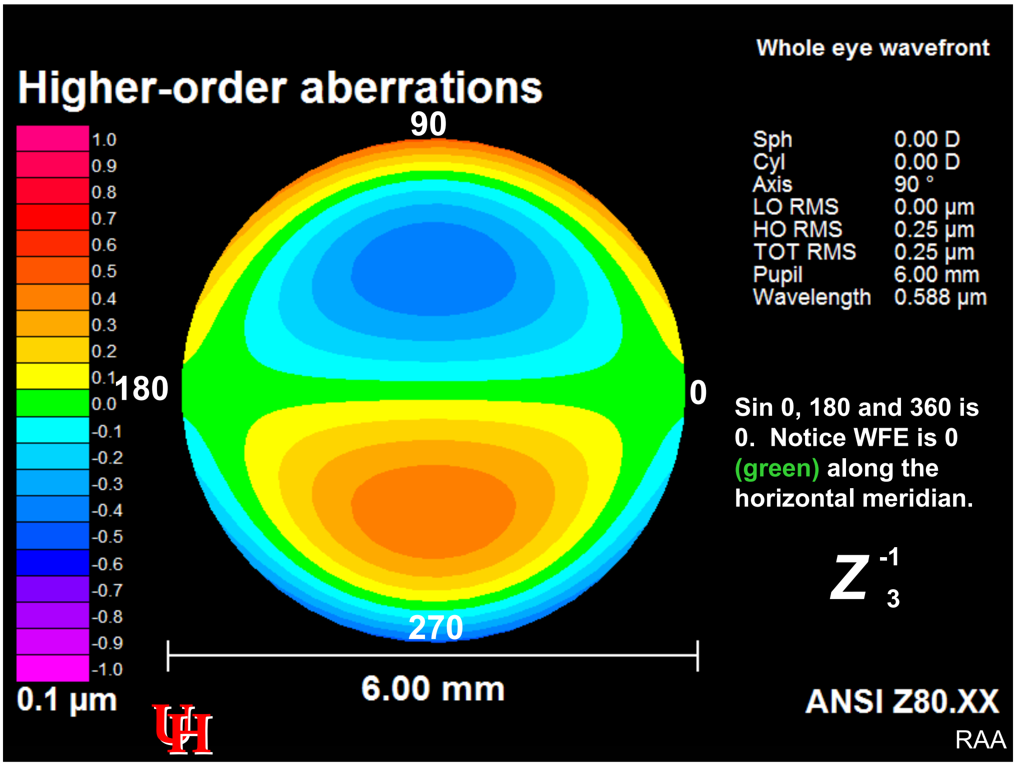
Notice that the sine of theta (1 × theta) has a positive portion between 0 and 180 and a negative portion as the angle theta varies between 180 and 360 degree.

Also notice the sine is zero at 0º, 180º and 360º degrees. That is the sine is zero along the entire horizontal meridian. If the sine of theta = 0 then the entire equation goes to 0. On the wavefront error map, green represent 0 WFE. Notice as dictated by the equation, the wavefront error is zero for the entire horizontal meridian (i.e, green along the entire horizontal meridian).

Notice also just like the sign of theta the map goes negative and positive once in 360º. Counting the number of full cycles for any individual mode will reveal the angular frequency. In the case above the map goes from warmer colors to cooler colors one time (i.e., one full cycle in 360º).

All Zernike coefficients with a negative superscript (remember the negative sign means that the angular frequency varies as the sin θ) will have a wavefront error of 0 along the horizontal meridian. See the next page for a wavefront map with a scale illustrating that cooler colors indicate positive (advanced) wavefront error and warmer colors negative (retarded) wavefront error.

It should be noted that it is not universally agreed that that warm colors should represent advanced wavefronts and cool colors retarded wavelengths. Assignment of color is currently being debated by the ANSI standards committee for the Specification of Ocular Wavefront Error. Hopefully a final decision will be made before the end of 2003.

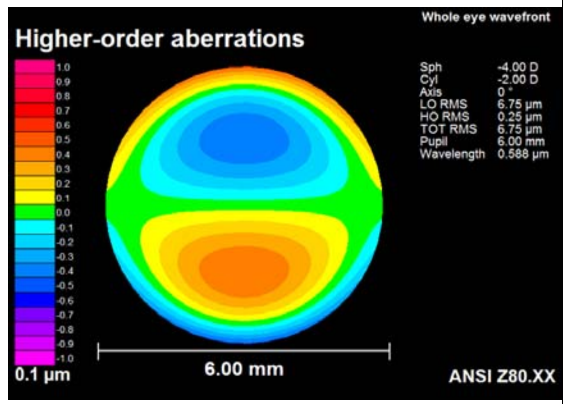


Advanced wavefront errors (i.e., wavefront errors that are in front of the reference wavefront) are warmer colors and have positive values. Retarded wavefront errors (i.e., wavefront errors that are behind the reference wavefront) are cooler colors and have negative values. Green represents 0 WFE. NOTE: This color scheme is currently being debated by the ANSI standards committee. A final answer will hopefully occur sometime in 2003.

Notice that the recommended map displays only higher order aberrations. The second order aberrations have been removed and are displayed as sphere, cylinder and axis values. In this case since the map is of vertical coma alone. The sphere, cylinder and axis values are all zero.

The reason for the standard map to display only the higher-order aberrations is that the major aberrations of the ametropic eye are the second order aberrations. As a consequence they generally swamp the lower order aberrations making them hard to impossible to see if included in the WFE map.

Put image here showing a 4 D myope with 2D astigmatism X 180 having .25 micrometers of vertical coma to illustrate how higher order aberrations can mask lower order aberrations



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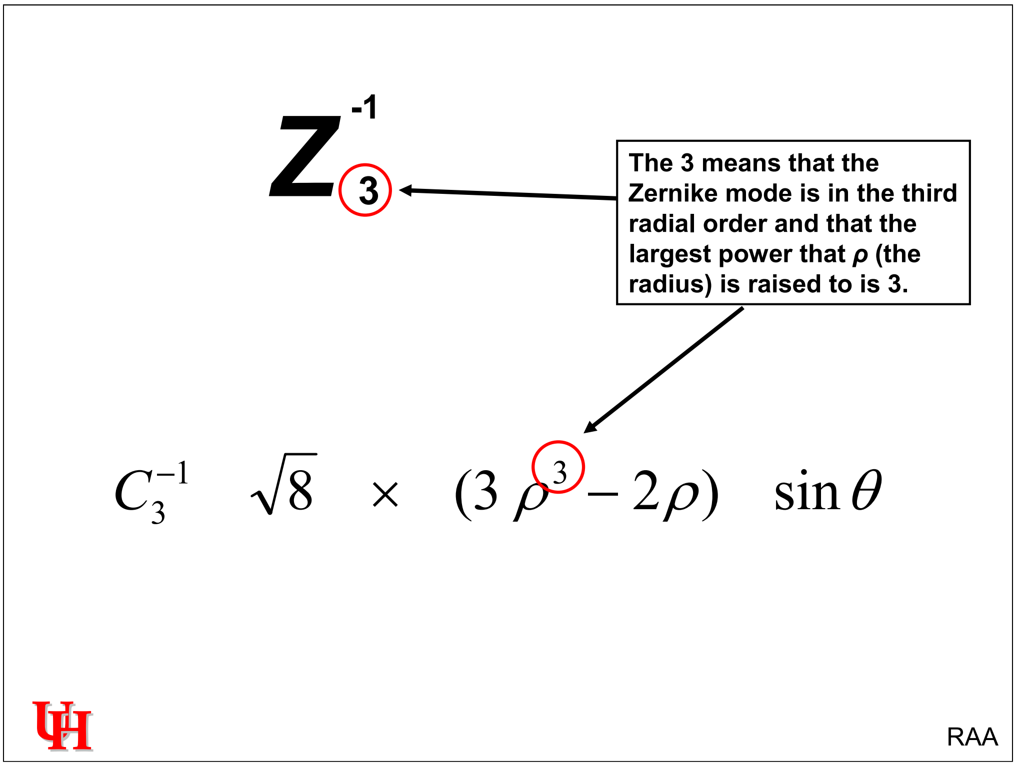
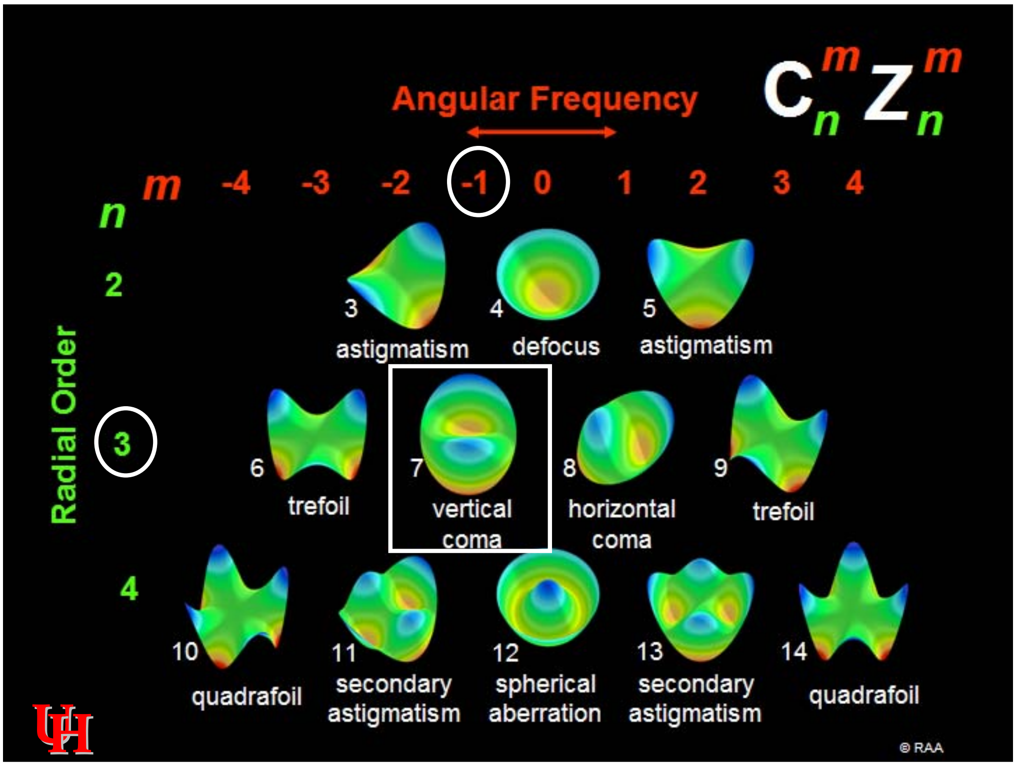
Notice on slide left that when the sphere and cylinder error is included in the wavefront map it masks the presence of the vertical coma and that the step size necessarily has to be large to display the large wavefront error contained in an eye needing -4.00 -2.00 × 180 correction. To see the effect of the higher order aberrations we need to remove the defocus and astigmatism errors (slide right). Notice the defocus and astigmatism terms are still included in the typed portion of the display in the traditional ophthalmic prescription format.

Further notice in this typed section that the low order aberrations (the traditional ophthalmic prescription) contributes 6.75 micrometers of RMS error over a 6mm pupil while the higher order aberration contributes only 0.25 micrometers of RMS error over a 6 mm pupil. It is worth emphasizing that when talking about RMS error it is critical to know the pupil size. To illustrate an RMS error of 0.25 micrometers over a 6 mm pupil loaded into defocus is a dioptric defocus of 0.19 D. A 0.25 micrometers of RMS defocus error over a 2 mm pupil is 1.73 D of defocus.

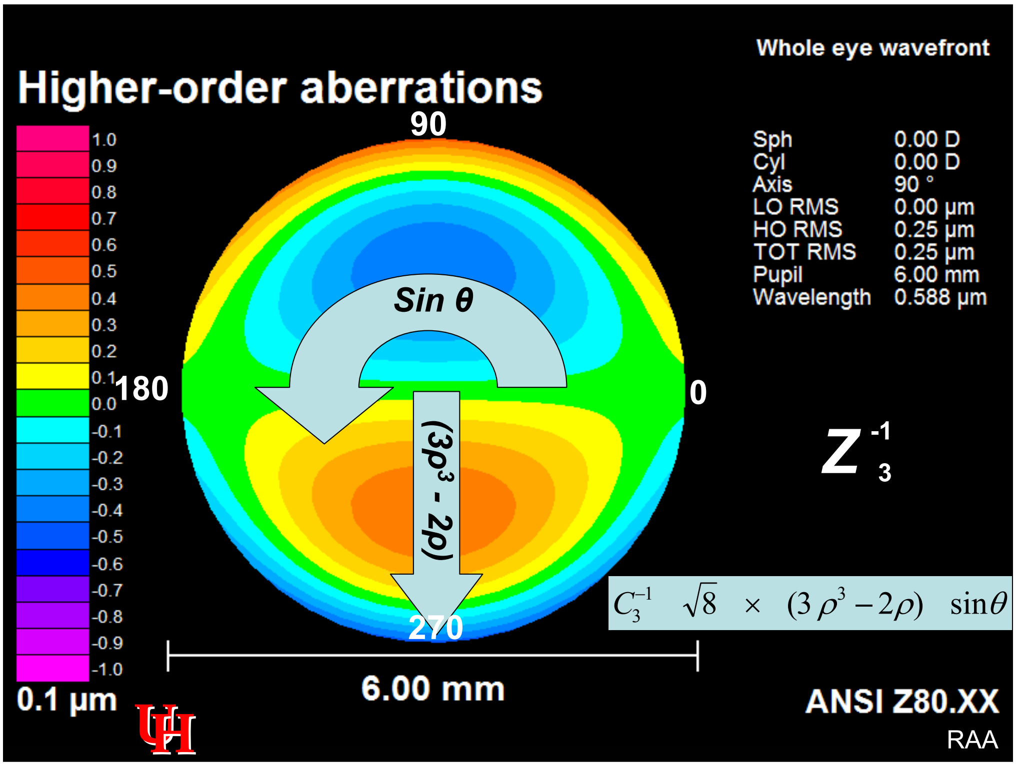


Back to the equation. The subscript designates the radial order and is the largest power that ρ (the radius) is raised to in the radial polynomial.

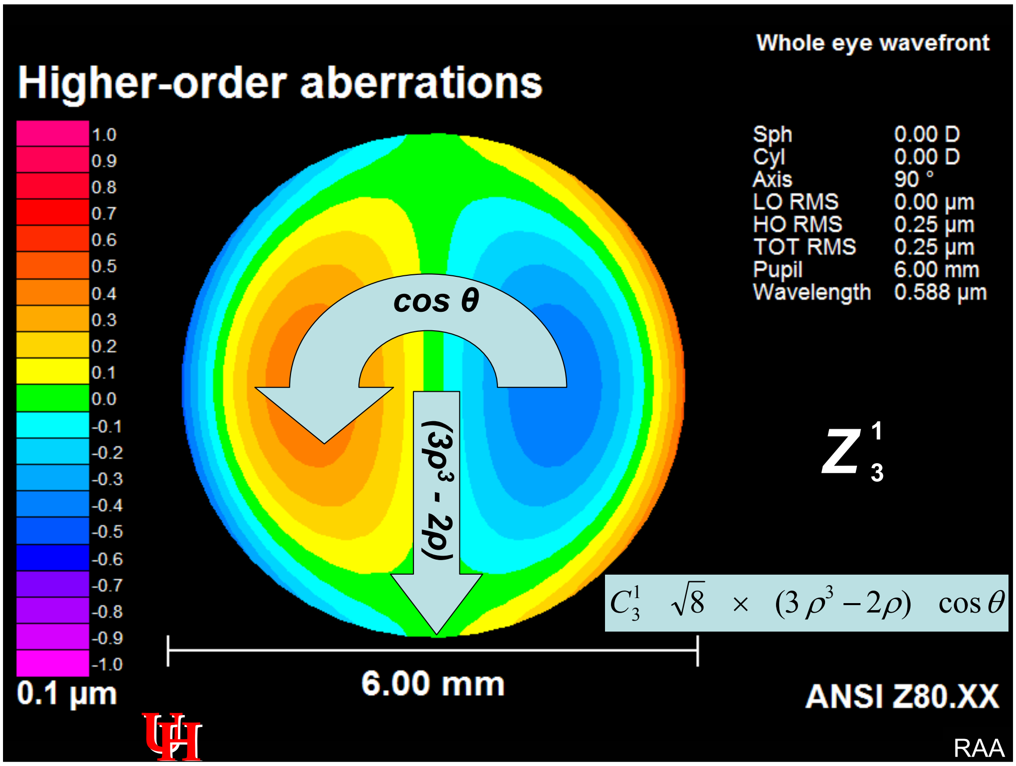
Hopefully it is becoming clear why the double index notation is the recommended notation for the Zernike expansion. Simply by looking at the values of the subscript and superscript we know the angular frequency of the wavefront error (thus how WFE for this mode varies as a function of angle) and how wavefront error is increasing with increasing radius.



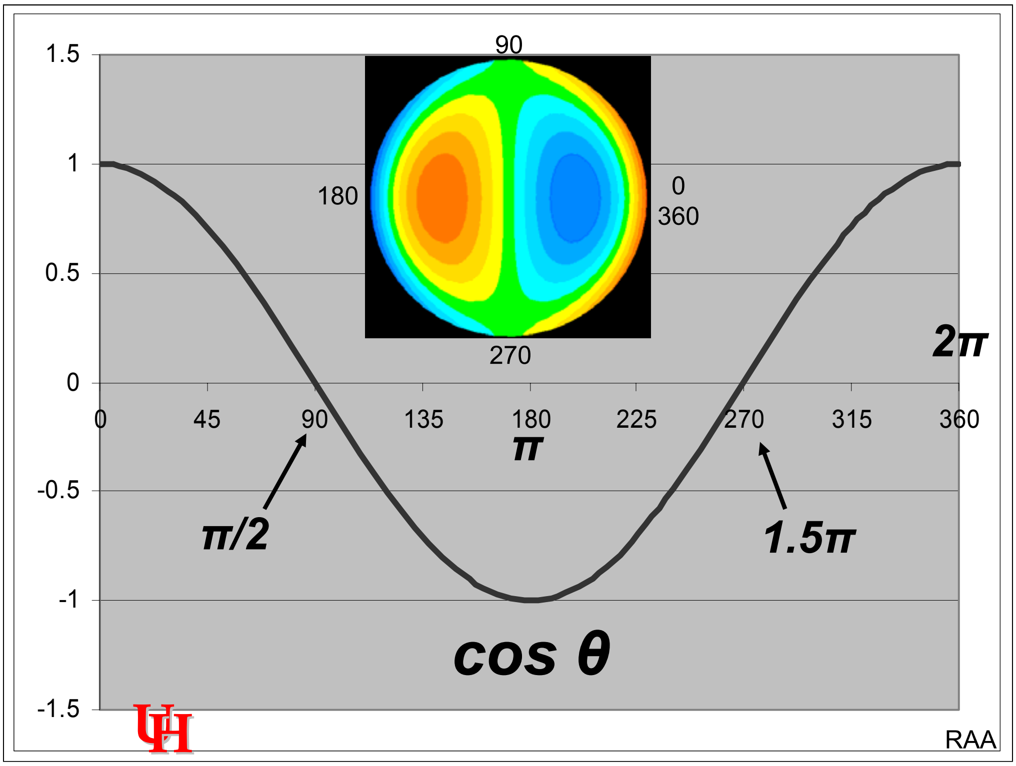
Looking back at our road map of the Zernike expansion if you examine the map more carefully you can see that vertical coma is in the third radial order and has an angular frequency that varies as sin 1θ.



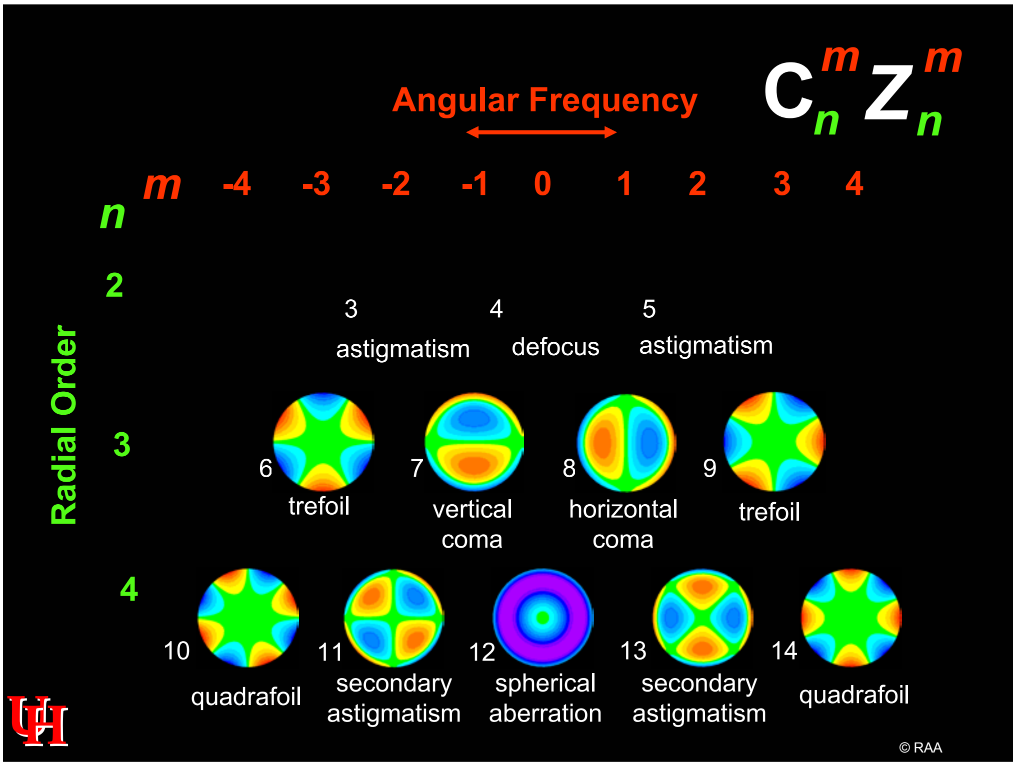
Use this slide to review.



Here we see the horizontal coma mode. Notice that the superscript is now a positive number meaning that angular frequency of the wavefront error varies with the cosine of *θ*. As a consequence the location of the advanced wavefront error (warmer colors) and retarded wavefront errors (cooler colors) have changed location within the pupil. This change in location is being dictated by cos *θ* .



Notice that the cosine is positive from 270º to 90º, negative from 90º to 270º, and zero at 90º and 270º. Notice that that when angular frequency is driven by the cos *θ,* the WFE is zero along a vertical meridian. However, as we will see in the next slide, unlike angular frequencies driven by the sin *θ* which lead to zero WFE along a horizontal meridian for all sine modes, angular frequency driven by the cos *θ* are zero along a vertical meridian when the angular frequency is an odd number.

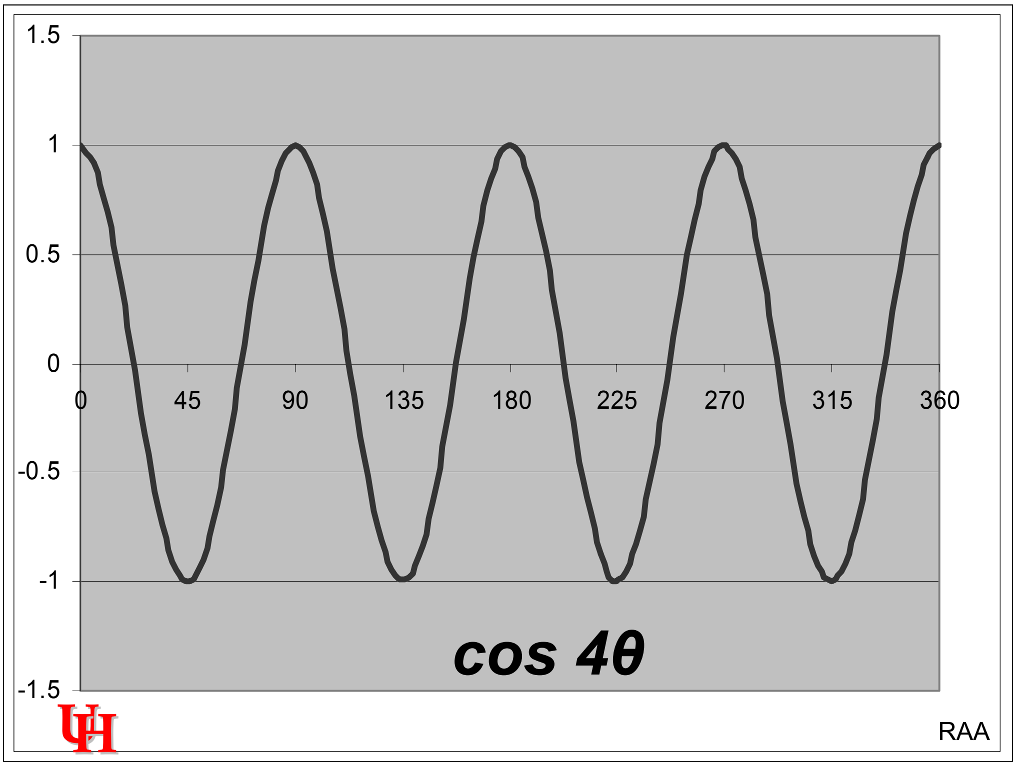


Notice that the WFE is 0 along the horizontal meridian for all radial orders driven by the sin θ (i.e., all angular frequencies with a minus designation). This is because when θ is 0º it does not mater how many times you multiply θ, it will always be 0 (i.e., 2 × 0º = 0º, 3 × 0º = 0º, 4 × 0º = 0º, etc,) and the sin of (0º) =0.

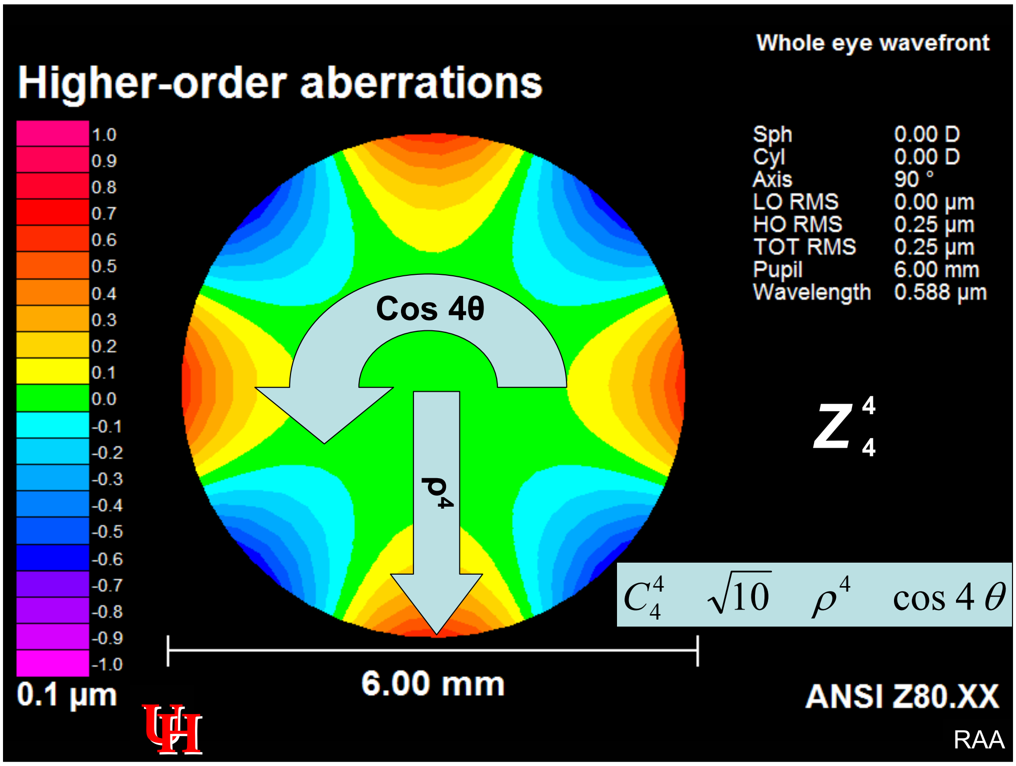
Notice that the WFE is 0 along the vertical meridian for all odd positive valued angular frequencies (driven by the cos θ). This is because θ is 0 at 90 and 270. When the angular frequency varies as a cosine, it does mater what the angular frequency is. That is, how many times θ is multiplied. If the multiple is an odd number, then WFE will be zero along a vertical meridian. If the multiple is an even number, then WFE will not be zero along a vertical meridian. The reason is simple, 2 × 90º = 180º, the cos 180º = 1, 3 × 90º = 270 º and the cos 270º = 0, 4 × 90º = 360º and the cos 360º = 1, etc.

Notice that it is easy to determine the angular frequency by simply counting the number of paired warm and cool oscillations in 360º. For example, both quadrafoil modes have 4 pairs of warm and cool oscillations. Not surprisingly the angular frequency is 4 for the quadrafoil modes. The next 2 slides more clearly illustrate this point for the cosine mode of quadrafoil.

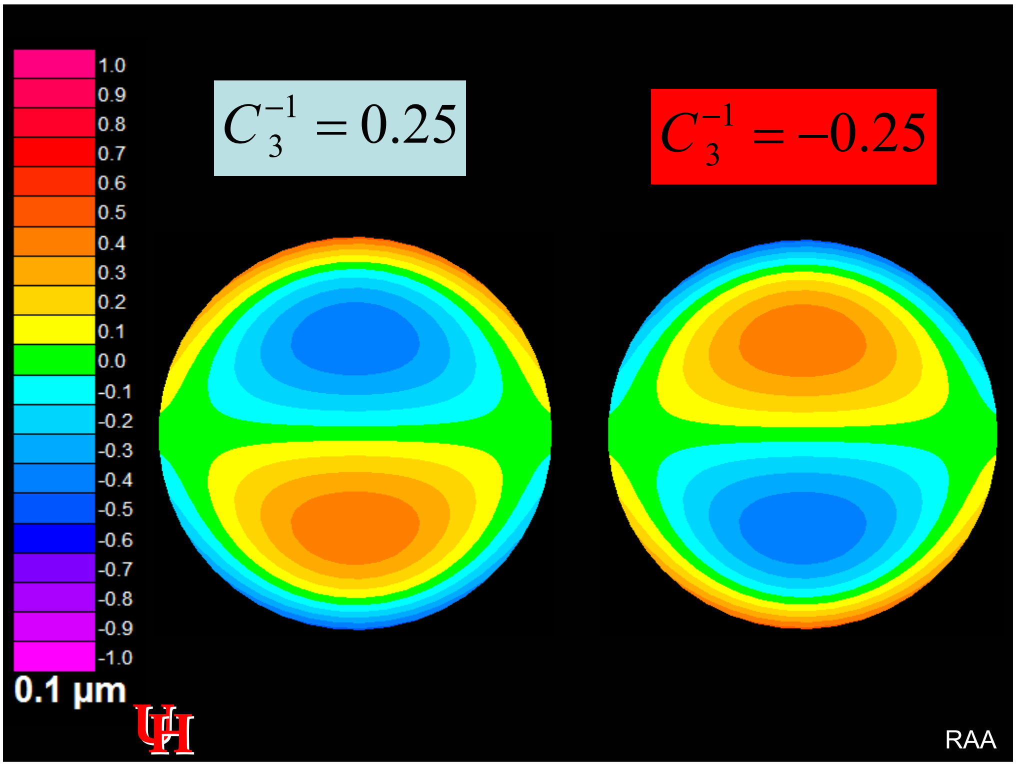
** Add modes 2, 4, and 5 once I can use CTView to plot these in the correct format.



If the angular frequency is 4 then there are 4 positive and 4 negative regions rotating through 360º.

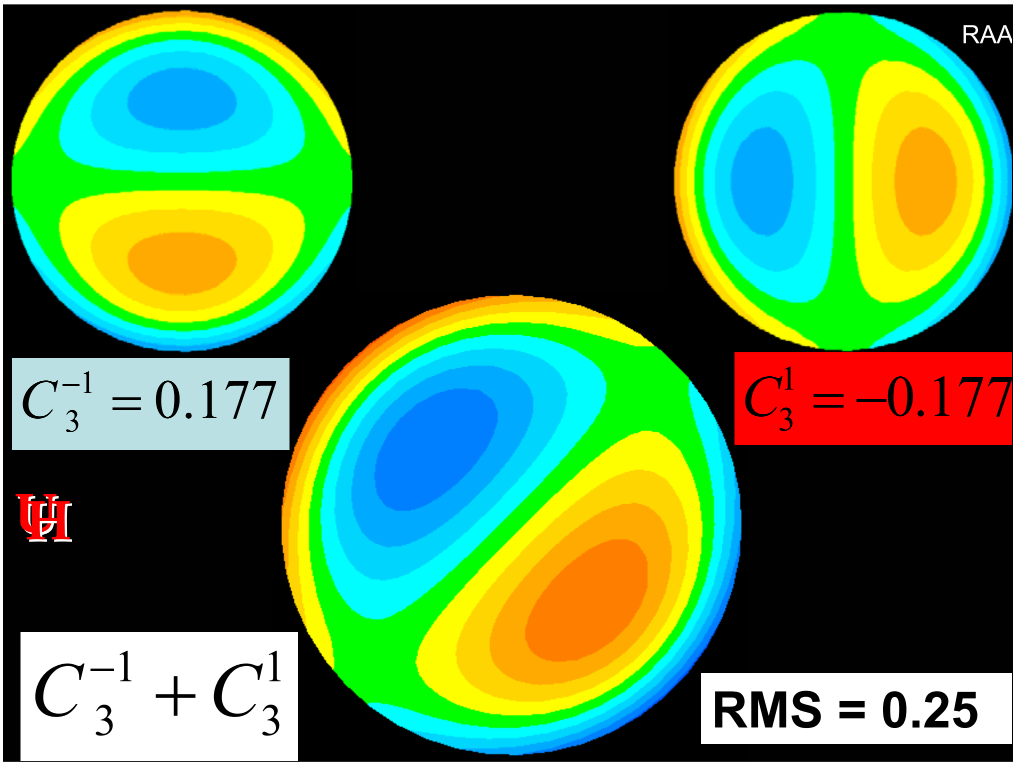


Notice that cos 4θ is positive 1 at 0º, 90º, 180º and 270º. Consequently, the wavefront is maximally advanced along these meridians at $\rho = 1$. Likewise cos 4θ is -1 at 45º, 135º, 225º and 315º (see previous slide). Consequently, the wavefront is maximally retarded along these meridians when $\rho = 1$. Remember the Zernike expansion is based on the unit circle. That is the maximum radius is 1. When plotting data for various pupil sizes the ρ value is multiplied by the actual pupil size.



Now let's explore what happens when the sign of the coefficient changes from positive to negative by looking at vertical coma and changing the sign of the coefficient.

Notice slide right when the coefficient becomes negative the warmer advanced portion of the map flips location and is now in the top portion of the graph and the cooler retarded portion of the wavefront is now in the lower portion of the graph.



The two coma terms can be added to obtain coma at any orientation. Here I have added equal portions of horizontal and vertical coma but of opposite sign. The result is coma with its orientation in the pupil rotated.

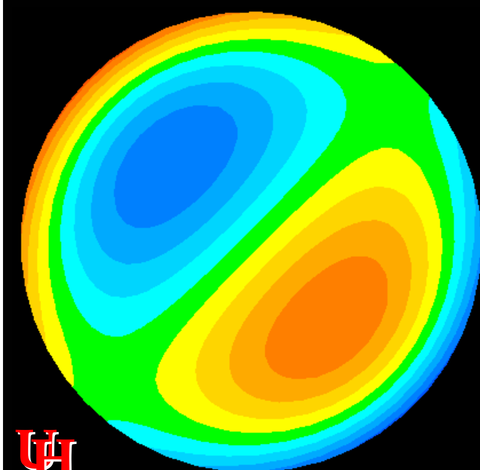
This figure also illustrates that total RMS error is not simply the sum of the absolute values of the underlying modes contributing to the total RMS error.

So what is the RMS error. RMS error is really the standard deviation of the WFE over the pupil. It is the calculated in the traditional method. That is it is the square root of the variance.

Let's look at this in greater detail in the next slide.

$$Variance = \text{the sum of the coefficients squared}$$

$$Variance = \sum_{n,m} {C_n^m}^2$$



$$RMS = \sqrt{\sum_{n,m} {C_n^m}^2}$$

$$RMS = \sqrt{0.177^2 + (-0.177)^2}$$

$$RMS = 0.25$$

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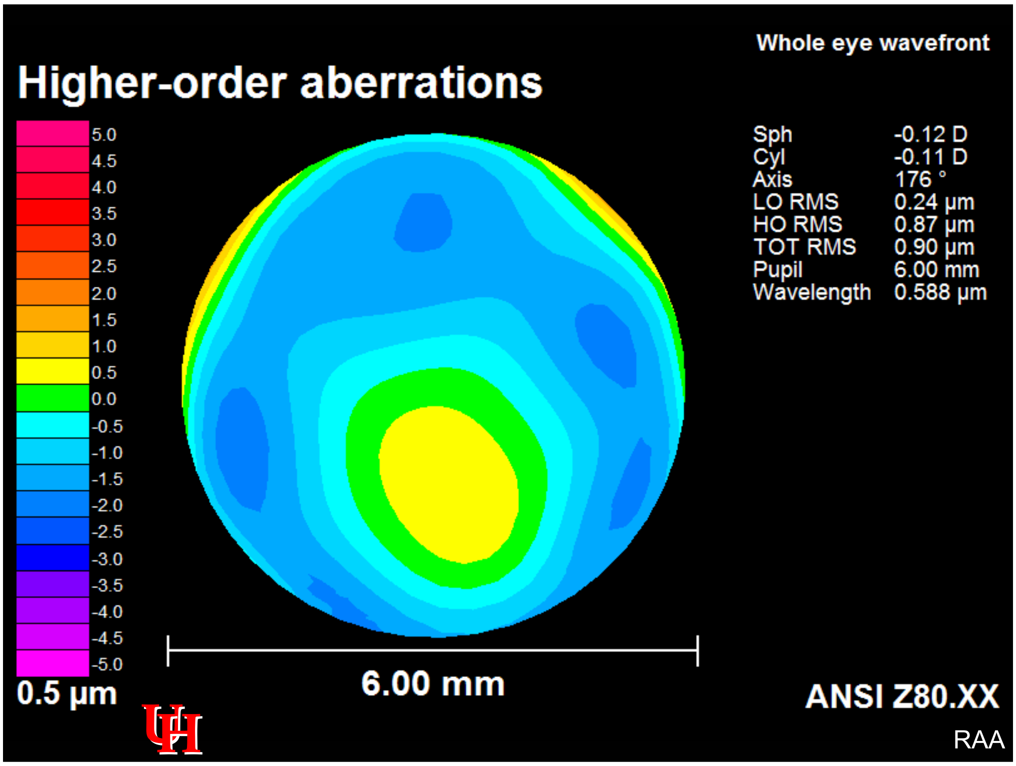
Because we have normalized the coefficients the variance is easily calculated. It is simply the sum of the coefficients squared.

RMS error equals the square root of the variance.



Now instead of looking at individual terms, let's look at the wavefront error of a real patient. This patient is a post LASIK patient.

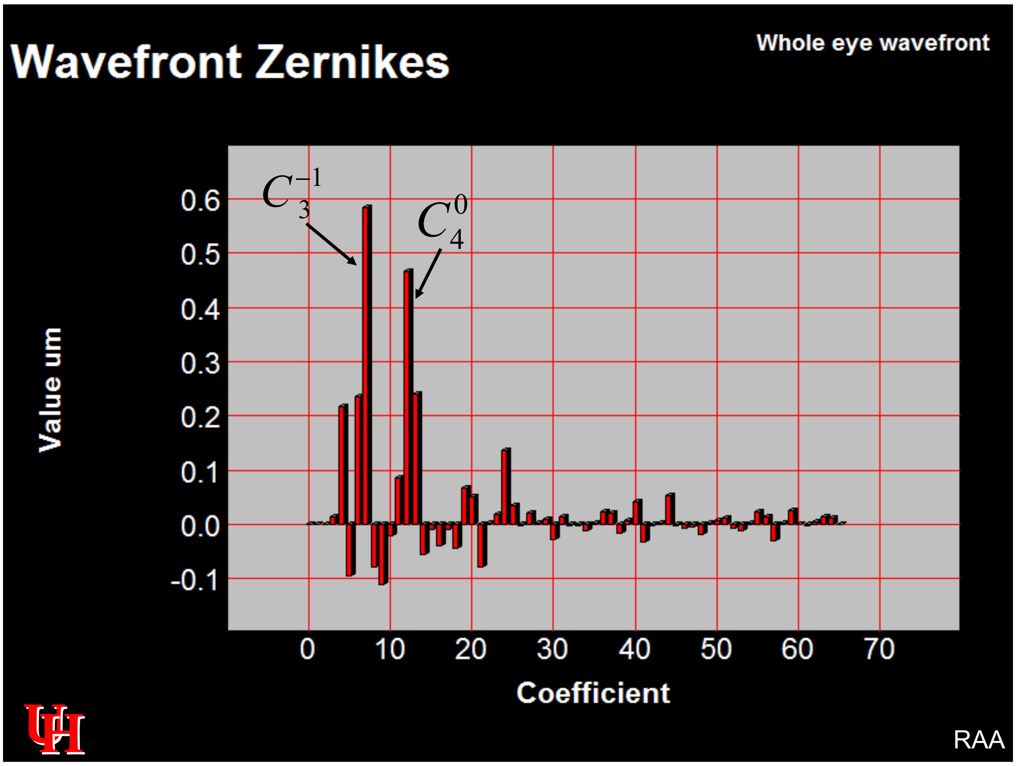
A real patient map is the linear sum of all Zernike modes used to fit the wavefront error.

How to look at the map.

First notice the pupil size. The wavefront error was measured over a 6mm pupil.

Second notice the step size is 0.5 microns. This is a fairly large step size for a higher order WFE map indicating that this eye has rather significant higher order aberrations. Notice the total higher order (HO) RMS is greater than the total lower order RMS due to sphere and cylinder.

Third, notice that the residual refractive error while not perfect is quite good (-0.12 – 0.11 × 176º) for a total lower order RMS of 0.24 micrometers. So if we were judging this outcome by residual spherical and cylindrical error alone, it would be an excellent outcome. If we judge the outcome using the residual higher order RMS WFE or residual total RMS WFE over a 6 mm pupil, the outcome is not nearly as good.



Looking at a graph of coefficient values of the Zernike modes through the 10th radial order we see that the two major contributors to the total RMS are term 7 (vertical coma) and term 12 (spherical aberration).

It is important to remember that while the individual terms of the Zernike expansion are mathematically independent. Their impact on vision is not. Individual aberrations do not have the same impact on visual performance (Applegate, RA, Sarver, EJ, Khemsara, V, Are All Aberrations Equal? Journal of Refractive Surgery, 18:S556-S562, 2002.) As aberrations increase within any one Zernike mode, the adverse consequence on visual performance increases (Applegate, RA, Ballentine, C, Gross, H, Sarver, EJ, Sarver, CA, Visual Acuity as a Function of Zernike Mode and Level of RMS Error, Optometry and Vision Science, Optometry and Vision Science, 80:97-105, 2003), and Zernike modes can interact to reduce the adverse visual impact or to increase the adverse visual impact (Applegate, RA, Marsack, J., Ramos, R., Interaction Between Aberrations Can Improve or Reduce Visual Performance, J Cataract and Refractive Surgery, in press).

I hope this has been helpful.

Please provide feedback to:

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