A Refined Bootstrap Method for Estimating the Zernike Polynomial Model Order for Corneal Surfaces

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***Abstract*—Following our previous work on optimal modeling of corneal surfaces with Zernike polynomials, we have developed a refined bootstrap-based procedure which improves the accuracy of the previous method. We show that for normal corneas, the optimal number of Zernike terms usually corresponds to the fourth or fifth radial order expansion of Zernike polynomials. On the other hand, for distorted corneas such as those encountered in keratoconus or in surgically altered cases, the estimated model was found to be up to three radial orders higher than for normal corneas.**

***Index Terms*—Cornea, model order selection, resampling techniques.**

I. Introduction

Modeling corneal surfaces with Zernike polynomials often leads to the question of the number of Zernike terms that should be used [1]. Recently, we have developed a bootstrap-based method to perform this task [2]. We have shown in simulations that the bootstrap method outperforms the classical model order selection techniques under the assumption that the measurement noise is independent and identically distributed (i.i.d.) across the whole corneal surface. In a further study [3], we have shown that this technique is appropriate in the context of fitting Zernike polynomials to corneal elevation data of normal subjects (i.e. subjects with healthy corneas), allowing judicious selection of the optimal number of Zernike terms.

However, for corneas with significant topographical deformities such as keratoconus, we have observed cases for which the residual error between the height data and the optimal fit as suggested by the bootstrap procedure is significantly larger than the one observed when testing videokeratoscopes with artificial surfaces. This implies that the model order is being underestimated. Other studies have also reported that a larger number of Zernike terms is expected for such corneas [4].

The shortcomings of the previously proposed bootstrap method can be attributed to the crucial assumption that the measurement noise is i.i.d.. We have conducted a thorough empirical analysis, which suggests that the measurement noise in corneal topography maps obtained from videokeratoscopes does not satisfy this requirement. In this communication we use the results of this empirical study to establish a model for the spatial distribution of the measurement noise and develop a bootstrap model selection procedure appropriate to this noise model.

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II. Measurement Noise in a Placido-Disk-Based Videokeratoscope

We used the Optikon 2000 Keratron videokeratoscope.[1] To evaluate the precision of the videokeratoscope and estimate the instrument's noise power, we have acquired a set of multiple measurements involving several artificial surfaces in two different instrument settings. The Keratron videokeratoscope has an embedded range finder for accurate estimation of the corneal apex and for automatic acquisition of corneal elevation.

In the first setting, the instrument was moved back and forth allowing activation of the range finder for every acquisition. In the second setting, the instrument was locked at a constant distance from the measured surface. The second setting was used in order to eliminate the noise related to the range-finder and establish the internal instrument's noise.

The maximum level of noise power did not exceed $5 \cdot 10^{-7}$ $[\mathrm{rad}^2]$ for slopes and $5 \cdot 10^{-1}$ $[\mu\mathrm{m}^2]$ for height data in the first setting and $5 \cdot 10^{-8}$ $[\mathrm{rad}^2]$ for slopes and $5 \cdot 10^{-2}$ $[\mu\mathrm{m}^2]$ for height data in the second setting. We noted that the noise power seemed to accurately fit a parabolic surface.

For real eyes, there are other factors that contribute to noise power [5]. We noted that for real corneas the noise power could reach several microns. However, the parabolic spatial distribution of noise is always evident. In the following section we develop a bootstrap algorithm which takes into account the observed spatial variations in the noise distribution. The assumption of independence is retained.

III. The Refined Bootstrap Algorithm

We model the anterior surfaces of the cornea with a finite series of Zernike polynomials as in [2]

$$C(\rho,\theta) = \sum_{p=1}^{P} a_p Z_p(\rho,\theta) + \varepsilon(\rho,\theta)$$

where $C(\rho,\theta)$ denotes the corneal surface, the index p is a polynomial-ordering number, $Z_p(\rho,\theta), p = 1,\ldots,P$, is the pth Zernike polynomial, a_p, $p = 1,\ldots,P$, is the coefficient associated with $Z_p(\rho,\theta)$, P is the order, ρ is the normalized distance from the origin, θ is the angle, and $\varepsilon(\rho,\theta)$ represents the measurement noise with distribution $F(\varepsilon;\rho)$ dependent on ρ. We assume that for a given ρ, $\varepsilon(\rho,\theta)$ is an i.i.d. random variable with zero-mean and finite variance σ_ρ^2.

The objective is to find the optimal model for the Zernike fit. Since the number of Zernike terms used for ophthalmic surfaces is relatively low, it is sufficient to find the optimal model order rather than a particular subset of the Zernike polynomials. Also, all rotationally asymmetric Zernike polynomial terms are paired and it is often desired to use both terms in a pair to determine the amplitude and axis of a particular aberration.

We have presented the bootstrap methodology and algorithm for finding the optimal number of Zernike terms in a fit to the corneal elevations in [2]. Herein, we provide only those details, which are related to the modified algorithm.

Algorithm

Step 1) Select an arbitrary model order $\beta_{\max} > P$, such that $\beta_{\max} < D$, where P is the true but unknown model order and D is the number of elevation samples, and find

[1]The authors do not have any proprietary or financial interests in the device mentioned.

the estimates of the coefficients $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{\beta_{\max}}\}$ of $\{a_1, a_2, \ldots, a_{\beta_{\max}}\}$ using the method of least-squares.

Step 2) Compute the surface estimate $\hat{C}(\rho_d, \theta_d) = \sum_{p=1}^{\beta_{\max}} \hat{a}_p Z_p(\rho_d, \theta_d)$.

Step 3) Calculate the residuals $\hat{r}_d = C(\rho_d, \theta_d) - \hat{C}(\rho_d, \theta_d)$, $d = 1, \ldots, D$.

Step 4) Sort the radial data into increasing order $\rho_{(1)} \leq \rho_{(2)} \leq \cdots \leq \rho_{(D)}$ and note the order index I_ρ. Derive an inverse index J_ρ of the index order I_ρ by sorting I_ρ into increasing order. The index J_ρ is used later to return to the original order of the data.

Step 5) Since the measurement noise power varies spatially, sort the residuals $\hat{r}_d$, $d = 1, \ldots, D$, using the order index I_ρ from Step 4) and form a vector $\hat{\boldsymbol{r}}_I = [\hat{r}_{I_\rho(1)} \;\; \hat{r}_{I_\rho(2)} \;\; \ldots \;\; \hat{r}_{I_\rho(D)}]$.
In this way, we can assume that the residuals $\hat{\boldsymbol{r}}_I$ are locally i.i.d.

Step 6) Select a block length M, and form a matrix by dividing $\hat{\boldsymbol{r}}_I$ into $N_b = D/M$ n-overlapping blocks

$$\hat{\boldsymbol{r}} = \begin{bmatrix} \hat{r}_{I_\rho(1)} & \hat{r}_{I_\rho(M+1)} & \cdots & \hat{r}_{I_\rho(D-M+1)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{r}_{I_\rho(M)} & \hat{r}_{I_\rho(2M)} & \cdots & \hat{r}_{I_\rho(D)}, \end{bmatrix} = [\hat{r}_{kl}]$$

$k = 1, \ldots, M$ and $l = 1, \ldots, N_b$.

We choose a block length M such that the variations in radial distance ρ in each of the columns of $\hat{\boldsymbol{r}}$ are sufficiently small.
Again, this leads to the assumption that the residuals in each of the columns of $\hat{\boldsymbol{r}}$ are i.i.d.

Step 7) For each column l in the matrix $\hat{\boldsymbol{r}}$, $l = 1, \ldots, N_b$, rescale the residuals $\tilde{r}_{kl} = \sqrt{M/L_M(l)}(\hat{r}_{kl} - M^{-1}\sum_{k=1}^{M} \hat{r}_{kl})$ and form a matrix of scaled residuals $\tilde{\boldsymbol{r}} = [\tilde{r}_{kl}]$, $k = 1, \ldots, M$ and $l = 1, \ldots, N_b$. Scaling of the residuals is performed for the bootstrap model order estimation to be consistent [2]. In our previous work, we have empirically estimated the global scaling pa rameter L_D for all the data [2]. However, to account for changes in spatial distribution of the noise, we will need to evaluate the scaling $L_M(l)$ for each column l in the matrix $\hat{\boldsymbol{r}}$, $l = 1, \ldots, N_b$

Steps 1)-7) can be considered as the preliminary steps of our procedure. In the following step, we perform the bootstrap model order selection.

Step 8) For all model orders $1 \leq \beta \leq \beta_{\max}$
a) calculate $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_\beta\}$ and $\hat{C}(\rho_d, \theta_d)$ as in steps 1 and 2.
b) For each column l in the matrix $\tilde{\boldsymbol{r}}$, $l = 1, \ldots, N_b$, draw independent bootstrap residuals $\tilde{r}^*_{kl}$ with replacement, from the empirical distribution of $\tilde{r}_{kl}$ and form a new matrix of residuals $\tilde{\boldsymbol{r}}^* = [\tilde{r}^*_{kl}]$, $k = 1, \ldots, M$ and $l = 1, \ldots, N_b$. Thus, the bootstrap resampling, which assumes i.i.d. data, is performed individually for each column of $\tilde{\boldsymbol{r}}$.
c) Rearrange the matrix $\tilde{\boldsymbol{r}}^*$ into a long vector using the inverse order index J_ρ

$$\tilde{\boldsymbol{r}}^*_J = [\tilde{r}^*_{J_\rho(1)} \quad \tilde{r}^*_{J_\rho(2)} \quad \ldots \quad \tilde{r}^*_{J_\rho(D)}] = [\tilde{r}^*_1 \quad \tilde{r}^*_2 \quad \ldots \quad \tilde{r}^*_D].$$

After this step, the bootstrap version of the residual error $\tilde{\boldsymbol{r}}^*_J$ corresponds spatially to the original measurement error and can be used for building new bootstrap surfaces.
d) Define the bootstrap surface $C^*(\rho_d, \theta_d) = \hat{C}(\rho_d, \theta_d) + \tilde{r}^*_d$.
e) Using $C^*(\rho_d, \theta_d)$ as the *new* surface, compute the least-squares estimate of $\{a_1, a_2, \ldots, a_\beta\}$, $\{\hat{a}^*_1, \hat{a}^*_2, \ldots, \hat{a}^*_\beta\}$, and calculate $\hat{C}^*(\rho_d, \theta_d) = \sum_{p=1}^{\beta} \hat{a}^*_p Z_p(\rho_d, \theta_d)$ and $\mathrm{SSE}^*_{D,M}(\beta) = D^{-1}\sum_{d=0}^{D}(C(\rho_d, \theta_d) - \hat{C}^*(\rho_d, \theta_d))^2$.
The mean square error $\mathrm{SSE}^*_{D,M}(\beta)$ for a given model order β depends on the block length M selected in step 6.
f) Repeat steps (b)-(e) a large number of times (usually more than a hundred) to obtain a total of B bootstrap statistics $\mathrm{SSE}^*_{D,M}(\beta)_1, \ldots, \mathrm{SSE}^*_{D,M}(\beta)_B$, and estimate the bootstrap mean-square error $\hat{\Gamma}_{D,M}(\beta) = B^{-1}\sum_{b=1}^{B} \mathrm{SSE}^*_{D,M}(\beta)_b$.
This is the main bootstrap resampling loop in which we obtain the bootstrap mean square error $\hat{\Gamma}_{D,M}(\beta)$ as a function of the model order β.

Step 9) In the last step, we choose that model order $\beta = \beta_o$ for which $\hat{\Gamma}_{D,M}(\beta)$ is a minimum.

To evaluate the performance of the proposed refined bootstrap procedure we simulated the same two surface models $C_1(\rho, \theta)$ and $C_2(\rho, \theta)$ as in [2]. The first surface represents Seidel's regular astigmatism with the model order $P_{C_1} = 5$, while the second surface represents horizontal coma with $P_{C_2} = 7$. Unlike in [2], the measurement noise

TABLE I
EMPIRICAL PROBABILITY (%) OF SELECTING THE MODEL ORDER FOR $C_1(\rho, \theta)$

β	$\hat{\Gamma}_{D,M}$	$\hat{\Gamma}_{D,LD}$	AIC	MDL	HQ	AIC_C
5	**99.5**	**86.7**	**37.0**	**86.0**	**25.2**	**45.3**
6	0.4	13.2	12.4	10.7	10.2	14.8
7	0.1	0.0	4.6	1.1	3.0	5.1
8	0.0	0.0	3.8	0.4	4.1	3.8
9	0.0	0.1	8.1	0.9	7.4	7.5
10	0.0	0.0	7.6	0.7	7.6	6.7
11	0.0	0.0	2.5	0.0	3.1	2.0
12	0.0	0.0	3.1	0.0	4.8	2.8
13	0.0	0.0	1.8	0.0	3.3	1.0
14	0.0	0.0	7.9	0.1	11.3	5.2
15	0.0	0.0	11.2	0.1	20.0	5.8

TABLE II
EMPIRICAL PROBABILITY (%) OF SELECTING THE MODEL ORDER FOR $C_2(\rho, \theta)$

β	$\hat{\Gamma}_{D,M}$	$\hat{\Gamma}_{D,LD}$	AIC	MDL	HQ	AIC_C
7	**99.4**	**86.2**	**41.3**	**85.9**	**28.2**	**50.3**
8	0.6	12.8	7.1	5.6	7.0	7.9
9	0.0	0.8	10.6	4.4	9.1	11.2
10	0.0	0.2	10.8	3.1	10.7	10.1
11	0.0	0.0	3.7	0.3	4.2	2.5
12	0.0	0.0	3.0	0.0	4.4	2.6
13	0.0	0.0	3.3	0.0	5.0	2.4
14	0.0	0.0	7.6	0.2	10.4	5.0
15	0.0	0.0	12.6	0.5	21.0	8.0

is modeled here as a zero-mean Gaussian noise process with spatially varying variance $\sigma_\rho^2 = \rho^2$. The performance of the bootstrap procedure in terms of the probability of selecting the correct model order was evaluated over 1000 independent realizations of the noise process $\varepsilon(\rho, \theta)$.

The results of this simulation are given in Tables I and II for surface model $C_1(\rho, \theta)$ and $C_2(\rho, \theta)$, respectively. Two major conclusions can be drawn from these results when compared to those of [2]. First, the performances of the classical methods of model order selection quickly deteriorate when the noise distribution spatially varies. Second, the refined bootstrap procedure overcomes the shortcomings of its predecessor and provides a very high probability of selecting the correct model order.

IV. EXPERIMENTAL RESULTS

Before the refined bootstrap procedure can be applied to model unknown corneal surfaces, the algorithm of Section III is calibrated. This is achieved by appropriately selecting $\beta_{\max}$, M, and L_M based on the measurement of known artificial surfaces. In our work with the Keratron videokeratoscope we chose, $\beta_{\max} = 45$ (corresponding to eighth radial order expansion), $M = 128$, and $L_M = (1/60)\rho + (1/30)$. B was set to 200 in all our experiments.

We have selected five subjects, each with significantly different corneas. Subject A had a healthy normal cornea with residual astigmatism of less than half of a diopter. Subject B had a healthy cornea with about two diopters of corneal astigmatism. Subject C had been diagnosed with early keratoconus. Subject D had advanced keratoconus while subject E had undergone a form of refractive surgery.

We have varied the corneal diameter from 2 mm to 8 mm for each of the subject's data to find the optimal model order of the Zernike expansion for different corneal regions. The results of estimating the model order for these subjects are shown in Fig. 1.

We note that the estimated optimal model order increases with the corneal diameter. For normal corneas, the optimal model order appears to be rather constant across the measured corneal surface. On the other hand, for keratoconics or surgically altered corneas, the optimal model order significantly increases for larger corneal diameters. Unlike the previous bootstrap method which underestimated the model order for deformed corneas, the proposed refined bootstrap method selects model orders in accordance with clinical expectations.

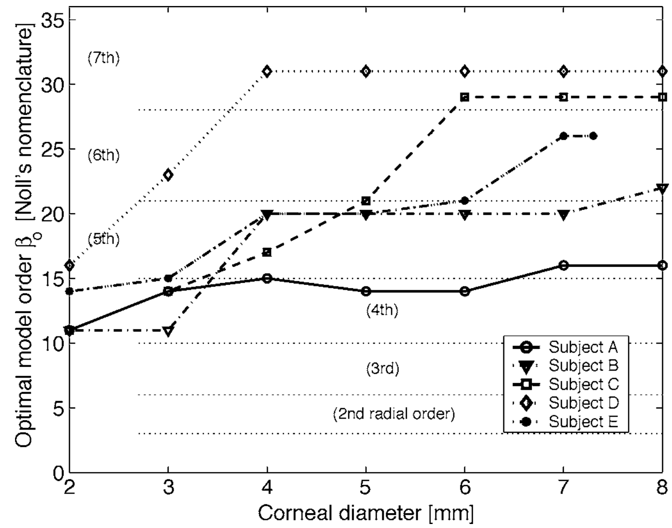

Fig. 1. Estimated model order selection as a function of corneal diameter. Subject A—normal cornea, B—astigmatic cornea, C—early keratoconus, D—advanced keratoconus, and E—post refractive surgery. The radial order bands of the Zernike polynomial expansion are indicated by horizontal dotted lines.

V. CONCLUSION

The results of fitting Zernike polynomials to corneal elevations with the proposed refined bootstrap algorithm are in a close agreement with our earlier experiences with artificial surfaces as well as with recently reported studies [4]. The higher Zernike polynomial model order required for distorted corneas was anticipated, but has not been statistically proven. The bootstrap is a powerful statistical method that appears suitable for this purpose. Also, the results indicate that fitting the same set of Zernike polynomials (say of the 8th order) to different corneas and different corneal radii will often result in overparameterization.

The proposed bootstrap method can be easily extended to other types of videokeratoscopes after analysis of the instrument's noise power is conducted. In cases where the noise distribution varies both radially and azimuthally, or when there is correlation between the noise samples, the bootstrap technique known as "block of blocks" could be used [6].

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