Chapter 3

Ordinary Differential Equations

Many engineering problems require the solution of ordinary differential equations (ODEs). In some cases these ODEs can be solved analytically to provide an exact solution. However it will often be the case that an analytic solution cannot be found (For example, consider the initial value problem $\frac{dy}{dx} = (1 + xy)^2, y(0) = 1$. This equation is non-separable and nonlinear which makes it difficult to solve). An approximate solution can still be generated using a numerical method.

In this chapter we will look it some numerical methods that can be used to solve ODEs. In particular we will consider first-order ODEs of the form $\frac{dy}{dx} = f(x, y)$. Higher-order ODEs can be written as a system of first-order ODEs so this approach can be applied to almost any ODE. In this chapter we will restrict our interest to initial value problems *i.e.,* cases where information about y (and its derivatives for higher-order problems) is known at some starting reference point x^0.

The numerical methods we will consider are all "step-by-step" methods. This means that we will start from the known solution y^0 at x^0 and proceed stepwise. In the first step we take a step of length h and estimate a solution y^1 at $x^1 = x^0 + h$. In the second step we use the value y^1 as a pseudo-initial value to estimate a solution y^2 at $x^2 = x^1 + h = x^0 + 2h$. Proceeding in this manner we will be able to construct an approximate solution for y over a specified range of x.

3.1 Euler's Method

3.1.1 Derivation

This method can be used to solve first-order ODEs of the form $\frac{dy}{dx} = f(x, y)$ with initial value $y(x^0) = y^0$.

All the methods we will consider in this chapter use a series of approximations to the derivatives in a Taylor's expansion over a finite region. Each of these methods matches the Taylor's expansion to some order. The omission of higher-order Taylor's series terms in the approximation means that each method will have an associated *truncation error*. The Euler method is the simplest of these methods in that it only approximates the first derivative of the function and this is simply done using a first-order finite difference approximation to the slope.

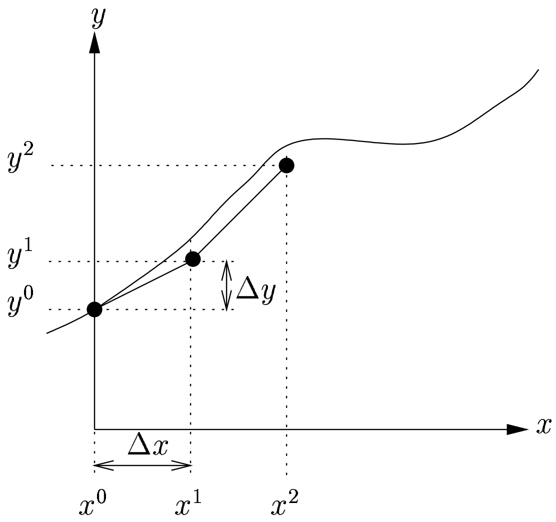

FIGURE 3.1: Steps in Euler's method.

The finite difference approximation to the derivative is:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y^{k+1} - y^k}{h} = f\left(x^k, y^k\right) \tag{3.1}$$

This can be rearranged to give

$$y^{k+1} = y^k + hf\left(x^k, y^k\right) \tag{3.2}$$

3.1.2 Implementation

For example consider the equation $\frac{dy}{dx} = (1 + xy)^2$ with $y(0) = 1$. Solving this equation using Euler's method with step length $h = 0.1$ gives:

k	x^k	y^k	$f\left(x^k, y^k\right) = \left(1 + x^k y^k\right)^2$	$y^{k+1} = y^k + hf\left(x^k, y^k\right)$
0	0	1	$(1 + (0)(1))^2 = 1$	$1 + 0.1 = 1.1$
1	0.1	1.1	$(1 + (0.1)(1.1))^2 = 1.2321$	$1.1 + (0.1)(1.2321) = 1.22321$
2	0.2	1.22321	$(1 + (0.2)(1.22321))^2 = 1.5491$	$1.22321 + (0.1)(1.5491) = 1.3781$
3	0.3	1.3781	$(1 + (0.3)(1.3781))^2 = 1.9978$	$1.3781 + (0.1)(1.9978) = 1.5779$
4	0.4	1.5779	*etc.*	

3.1.3 Error Estimation

If $y\left(x^0+h\right)$ is the exact value at x^0+h then, by Taylor's series:

$$\begin{aligned} y\left(x^0+h\right) &= y\left(x^0\right)+hy'\left(x^0\right)+\frac{h^2}{2}y''\left(x^0\right)+\ldots \\ &= y\left(x^0\right)+hf\left(x^0,y^0\right)+\frac{h^2}{2}y''\left(x^0\right)+\ldots \\ &= \text{Euler's Method estimate} + \mathrm{O}\left(h^2\right) \end{aligned} \tag{3.3}$$

Therefore the error in one step is $\mathrm{O}\left(h^2\right)$ which means the truncation error involves terms which are proportional to h^2 or higher powers.

If we halve the step length then the error per step is proportional to $\left(\frac{h}{2}\right)^2 = \frac{h^2}{4}$ but to get to the same point we must do twice as many steps which means the error at the final point is $E \propto 2\left(\frac{h^2}{4}\right) = \frac{h^2}{2}$. Therefore by halving the step length we halve the error. Similarly if we double the step length we double the error. The error for Euler's method is proportional to the step length. For this reason Euler's method is termed a first-order method. In general if a method is said to be n-th order it will have a truncation error of $\mathrm{O}\left(h^{n+1}\right)$.

3.1.4 Stability

The Euler method is only a first-order accurate method. Therefore in practice it is generally necessary to use small step lengths to achieve stable and accurate solutions. For example, consider the equation $y' = ay$. The solution to this equation is $y = e^{ax}$ which will be a decaying exponential for $a < 0$. Applying the Euler method to this equation gives:

$$\begin{aligned} y^1 &= (1+ha)\,y^0 \\ y^2 &= (1+ha)\,y^1 = (1+ha)^2\,y^0 \\ y^k &= (1+ha)^k\,y^0 \end{aligned} \tag{3.4}$$

Therefore if $|1+ha| > 1$ then y^k will increase in value. But if $a < 0$ we want y^k to decrease. This implies that (for $a < 0$) we require $-1 < 1+ha < 1 \longrightarrow 1+ha > -1 \longrightarrow ha > -2 \longrightarrow h < -\frac{2}{a}$. Therefore the stability criterion for the Euler method to give stable solutions for this problem is $h < -\frac{2}{a}$.

3.2 The Improved Euler Method

3.2.1 Derivation

Euler's method uses the slope (*i.e.*, the value of y') at y^k to predict y^{k+1}. If the slope is changing significantly over the step length then Euler's method will give an inaccurate solution. An improved method can be devised by averaging the slopes at y^k and y^{k+1}.

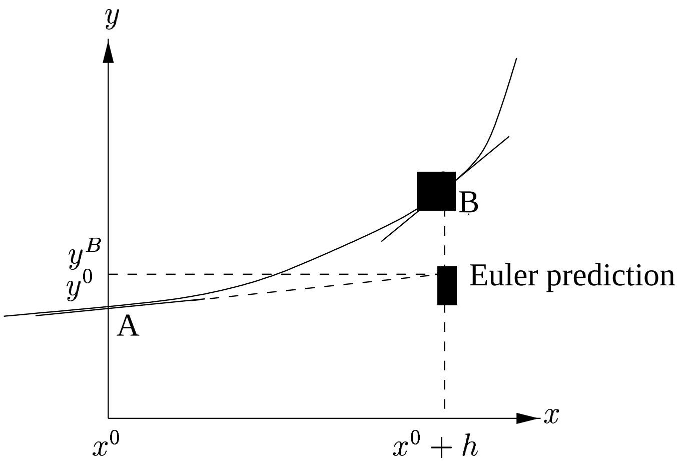

FIGURE 3.2: Euler prediction for the improved Euler method

Consider Figure 3.2. At A the slope is given by $y' = f\left(x^0, y^0\right)$. At B the slope can be estimated as $y' = f\left(x^0 + h, y^B\right)$ where using an Euler estimation $y^B = y^0 + hf\left(x^0, y^0\right)$.

Now instead of using just the slope at A to obtain y^{k+1} we can improve the Euler prediction by using the average slope from A and B as shown in Figure 3.3.

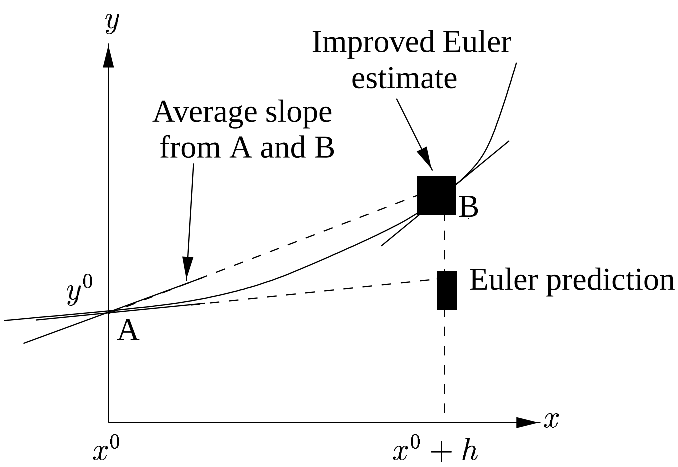

FIGURE 3.3: Improved Euler estimate using the average slope at the current point and the Euler predicted point.

Hence for the improved Euler method

$$\begin{aligned} y^1 &= y^0 + h \times \text{average slope} \\ &= y^0 + \frac{h}{2}\left[f\left(x^0, y^0\right) + f\left(x^0 + h, y^B\right)\right] \end{aligned} \tag{3.5}$$

The Improved Euler Method uses this approach to approximate the solution at y^{k+1} as:

$$y^{k+1} = y^k + \frac{h}{2}\left[f\left(x^k, y^k\right) + f\left(x^{k+1}, y^k + hf\left(x^k, y^k\right)\right)\right] \tag{3.6}$$

The Improved Euler Method is an example of a predictor-corrector method. In this first step the value y_B is predicted using an Euler step. This allows extra slope information to be determined which can be used to make a corrected estimate of y^{k+1}.

The Improved Euler Method is a second-order method *i.e.*, truncation error $\sim \mathrm{O}\left(h^3\right)$. The Improved Euler Method is more computationally expensive than Euler's Method but it provides more accurate solutions.

3.2.2 Implementation

For example consider the equation $\dfrac{dy}{dx} = (1 + xy)^2$ with $y(0) = 1$. Solving this equation using the Improved Euler Method with step length $h = 0.1$ gives:

k	x^k	y^k	f^k	y_E^{k+1}	f_E^{k+1}	y^{k+1}
0	0	1	1	1.1	$(1+0.1(1.1))^2 = 1.2321$	$1 + \frac{0.1}{2}(1 + 1.2321) = 1.1116$
1	0.1	1.1116	1.2347	1.2351	1.5551	1.2511
2	0.2	1.2511	*etc.*			

where $f^k = \left(1 + x^k y^k\right)^2$, $y_E^{k+1} = y^k + hf^k$, $f_E^{k+1} = \left(1 + x^{k+1} y^{k+1}\right)^2$ and $y^{k+1} = y^k + \frac{h}{2}\left(f^k + f_E^{k+1}\right)$.